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# Lattice QCD and Kaon Physics

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Kaon 2013  
University of Michigan April 29, 2013

# Introduction

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Kaon physics has become a precision science.

I will cover:

- Introduction to Lattice QCD
- light quark masses and decay constants
- $K \rightarrow \pi l \nu$  form factors
- kaon mixing:  $B_K$
- $K \rightarrow \pi\pi$  decays and  $\Delta M_K$

# Why Lattice QCD?

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The QCD coupling  $\alpha_s$  runs with distance scale. At long distances, the theory is strongly coupled.

We need a reliable nonperturbative tool to calculate all the low energy phenomena of QCD, from the hadron spectrum to quark masses to weak matrix elements.

Precision determination of weak matrix elements is required to make best use of existing experimental data from B-factories, CLEO, LHCb, etc. Most indirect physics searches require nonperturbative input.

# Lattice QCD in the LHC era

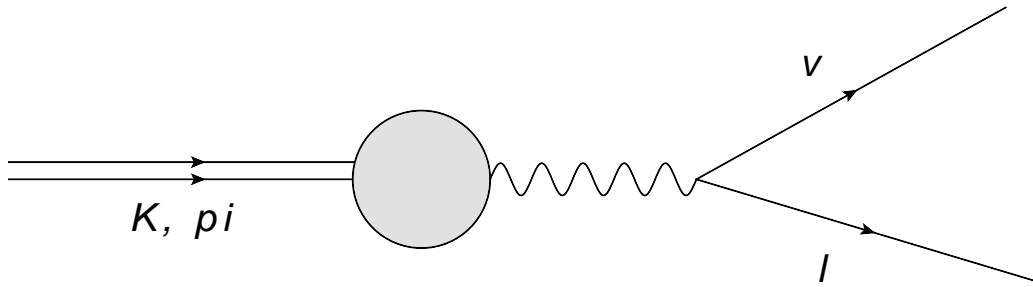
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If a “particle zoo” is discovered, ATLAS and CMS will measure the spectrum. Precision flavor measurements still important as part of studies to learn the underlying structure of the theory.

If new physics is beyond the reach of direct production at LHC, indirect searches using high precision low energy quantities may be our best bet to discover new physics.

# Nonperturbative input needed

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$$\Gamma = (\text{known factor}) (\text{CKM factor}) (\text{QCD factor}) \quad (1)$$

$$\frac{\Gamma(K \rightarrow l\bar{\nu}_l)}{\Gamma(\pi \rightarrow l\bar{\nu}_l)} = \left( \frac{|V_{us}|}{|V_{ud}|} \right)^2 \left( \frac{f_K}{f_\pi} \right)^2 \frac{m_K \left( 1 - \frac{m_l^2}{m_K^2} \right)^2}{m_\pi \left( 1 - \frac{m_l^2}{m_\pi^2} \right)^2} \left[ 1 + \frac{\alpha}{\pi} (C_K - C_\pi) \right] \quad (2)$$

# Lattice QCD Calculations

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Calculate expectation values on an ensemble of gauge fields  $[\mathcal{U}]$  with an exponential weight

$$\langle \mathcal{O} \rangle = \frac{1}{Z} \int D\mathcal{U} D\psi_{\text{sea}} D\bar{\psi}_{\text{sea}} e^{-S_{\text{QCD}}[\mathcal{U}, \psi_{\text{sea}}, \bar{\psi}_{\text{sea}}]} \mathcal{O}[\mathcal{U}, \psi_{\text{val}}, \bar{\psi}_{\text{val}}], \quad (3)$$

$$\langle \mathcal{O} \rangle = \frac{1}{Z} \int D\mathcal{U} \prod_{f=1}^{n_f} \det(\mathcal{D} + m_f) e^{-S_{\text{QCD}}[\mathcal{U}]} \mathcal{O}[\mathcal{U}, \psi_{\text{val}}, \bar{\psi}_{\text{val}}], \quad (4)$$

The action is discretized, so that derivatives become finite differences. Integral is still too large to do directly ( $N_s^3 \times N_t \times 4 \times N_f \times N_c$ ), so we use Monte Carlo importance sampling.

# Computing

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# Types of Errors

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Because QCD with physical quark masses is a nonlinear multiscale problem ( $\Lambda_{QCD} \approx 100 - 200 \text{ MeV}$ ,  $m_{u,d} \approx 2 - 6 \text{ MeV}$ ,  $m_b \approx 4.3 \text{ GeV}$ ), it is very expensive to simulate at the physical quark masses.

- 1.) Statistics and fitting
- 2.) Tuning lattice spacing,  $a$ , and quark masses
- 3.) Matching lattice gauge theory to continuum QCD
- 4.) Extrapolation to continuum
- 5.) Chiral extrapolation to physical up, down quark masses
- 6.) **Quenching. Uncontrolled!**



# Quenched Approximation

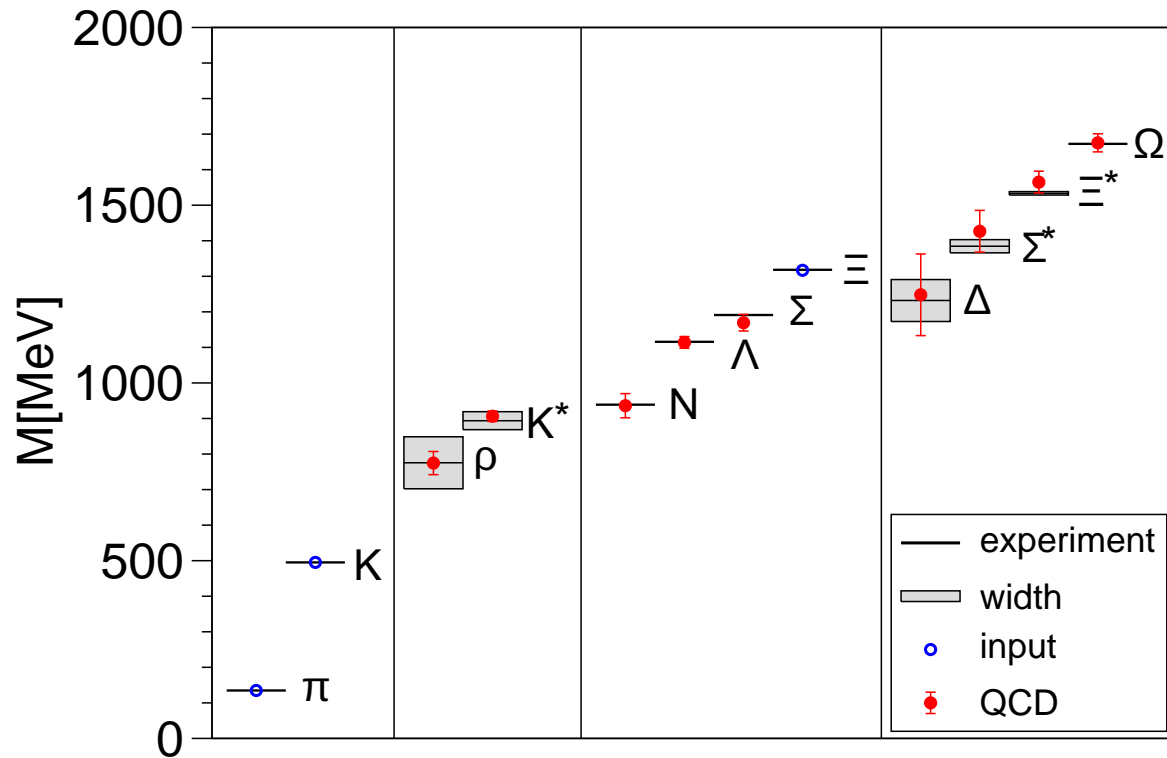
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Configurations are generated with a weighting given by the gauge field and fermion determinant. Including the fermion determinant in this weighting is the most computationally demanding step in lattice QCD.

The quenched approximation ignores fermion-antifermion vacuum bubbles. This is an uncontrolled systematic error.

“Unquenched” calculations, where the **fermion determinant** is included, are now the norm.

# Hadron Spectrum from Lattice QCD



BMW Collaboration, Science 322:1224-1227, 2008.

🟢 *Good agreement between theory and experiment!*

# Quenching the strange quark

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- Strange threshold lies in the nonperturbative regime.
- Most quantities show no difference with 2+1 flavor results, with precision at the 3-5% level.
- Error is difficult to estimate. Could be as much as 5%, and only sure way to quantify it is to compare with 2+1 flavor results.
- Since the lattice world averages are approaching the level where quenching the strange is likely to be important, we do not include 2 flavor results in our averages.

# Some types of lattice fermions

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- **Wilson Fermions:** Introduces an additional “irrelevant” term to the action. Hard breaking of chiral symmetry.
- **Staggered Fermions:** Identifies some of the extra fermions with the 4 different spin components of a single fermion. This brings us down to 4 extra fermions. This factor of 4 is eliminated by taking the 4th root of the fermion determinant. Some open theoretical issues with this, though theoretical progress has been made on this front.
- **Domain Wall Fermions:** Solves chiral symmetry problem by using Wilson type quarks in five dimensions. More costly because of the extra dimension. There is a small chiral symmetry breaking due to the finiteness of the fifth dimension.
- **Overlap Fermions:** Exact lattice chiral symmetry. Very expensive.

# Some Ongoing Lattice Projects

Group	$N_f$	action	$a(\text{fm})$	$m_\pi L$	$m_\pi^{\min}$ (MeV) sea/val
ETMC	2(+1+1)	Twisted Mass	0.05-0.10 fm	$\gg 1$	280/280
MILC	2+1(+1)	staggered	0.045-0.12 fm	$> 4$	130/140
RBC/UKQCD	2+1	Domain Wall	0.085-0.15 fm	$> 4$	180/135
JLQCD	2+1	Overlap	0.11 fm	$\geq 2.7$	310/310
PACS-CS	2+1	Clover	0.09 fm	$\geq 2.0$	140/140
BMW	2+1	Clover	0.054-0.125 fm	$\geq 4$	135/135
HPQCD	2+1(+1)	staggered	0.045-0.15 fm	$> 4$	130/170

(In staggered calculations, the sea pion mass quoted is the rms value. The valence pion mass quoted is the taste-goldstone.)

# Lattice Averages

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[www.latticeaverages.org](http://www.latticeaverages.org) for relatively recent updates of lattice averages based on JL, E Lunghi, R S Van de Water, Phys Rev D 81 034503 (2010) [arXiv:0910.2928]. FLAG(Flavor Lattice Averaging Group) also produced averages [EPJ C71:1695 (2011)]. Methodology differs somewhat, results are broadly consistent. The two groups are merging into FLAG II.

Includes light and heavy quark physics quantities, many weak-matrix elements for flavor physics.

Criteria for FLAG II:

- Only quantities that are documented in publications are included.
- Only quantities that include complete statistical and systematic error budgets are included in the averages.
- Averages only include 3 (or 4) flavor numbers. It is difficult to assess the error due to quenching the strange quark, and the  $\sim$ percent level precision for some averages is approaching the size that one would expect for this effect.

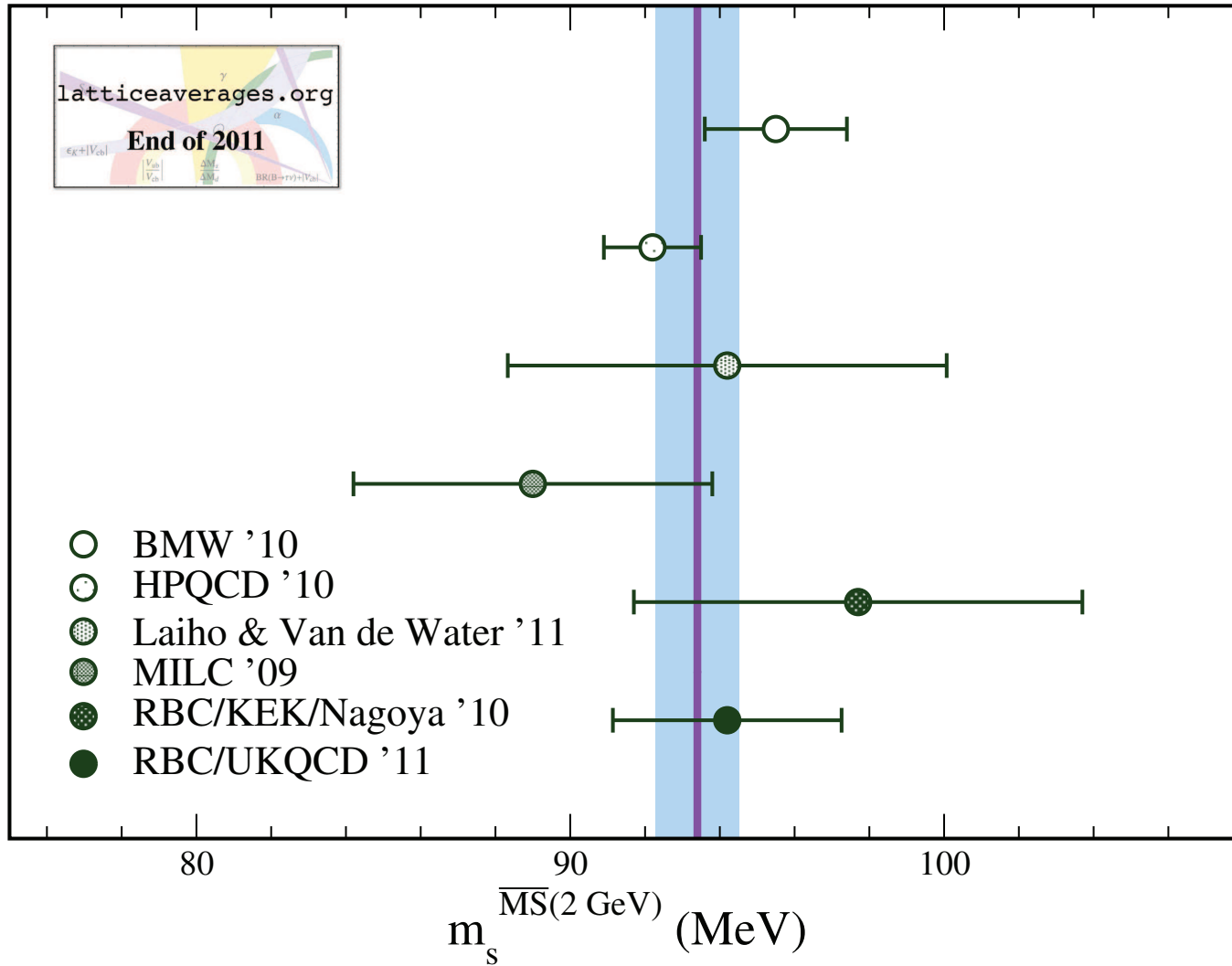
Coming soon!

# FLAG II members

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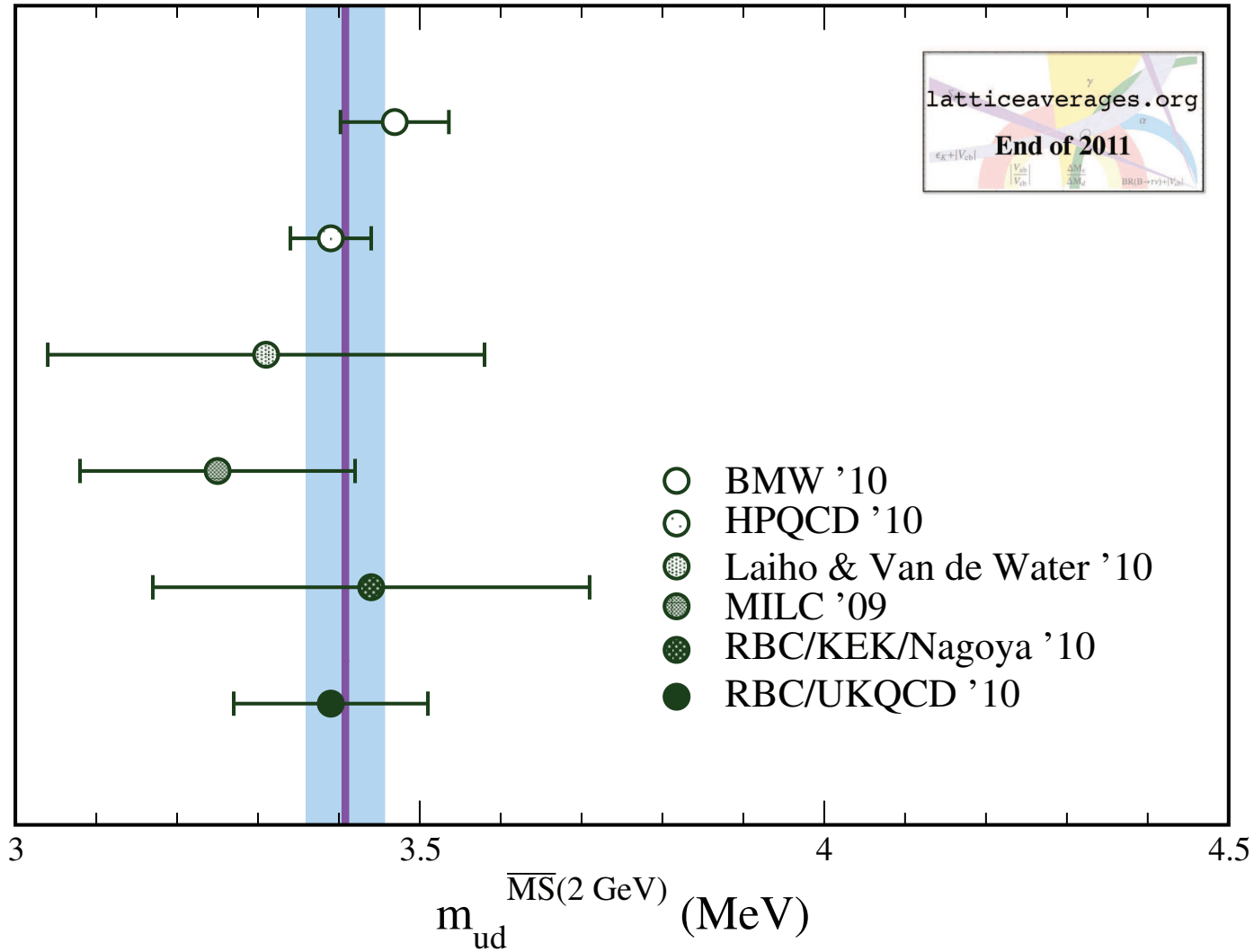
- Advisory Board: S. Aoki, C. Bernard, C. Sachrajda
- Editorial Board: G. Colangelo, H. Leutwyler, T. Vladikas, U. Wenger
- Working Groups:
  - **Quark masses**: L. Lellouch, T. Blum, V. Lubicz
  - $V_{us}, V_{ud}$ : A. Jüttner, T. Kaneko, S. Simula
  - **LEC's**: S. Dürr, H. Fukaya, S. Necco
  - $B_K$ : H. Wittig, JL, S. Sharpe
  - $\alpha_s$ : R. Sommer, T. Onogi, R. Horsley
  - $f_B, B_B$ : A. El Khadra, Y. Aoki, M. Della Morte
  - **$B$  and  $D$  semileptonic form factors**: R. Van de Water, E. Lunghi, C. Pena, J. Shigemitsu

# Strange quark mass

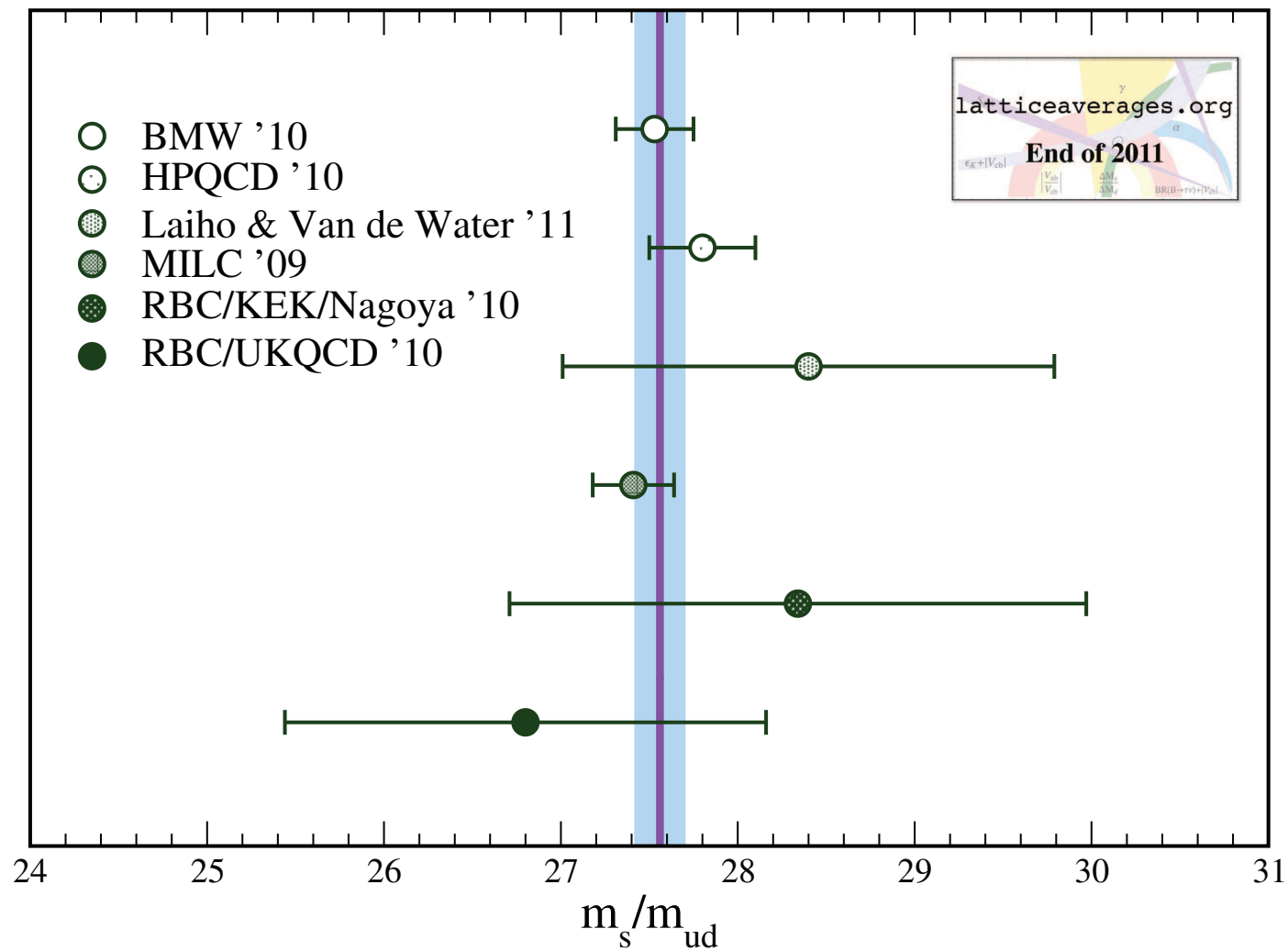




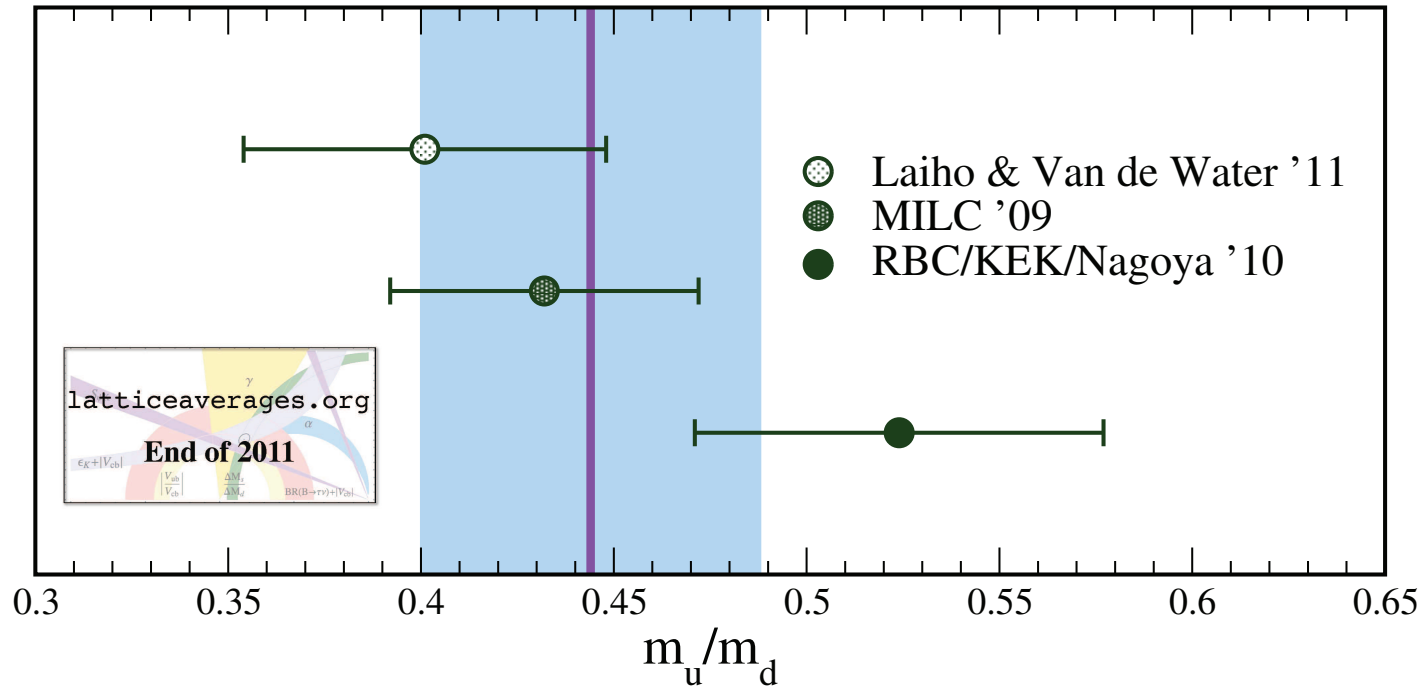
# Light-quark mass



# Quark mass ratio

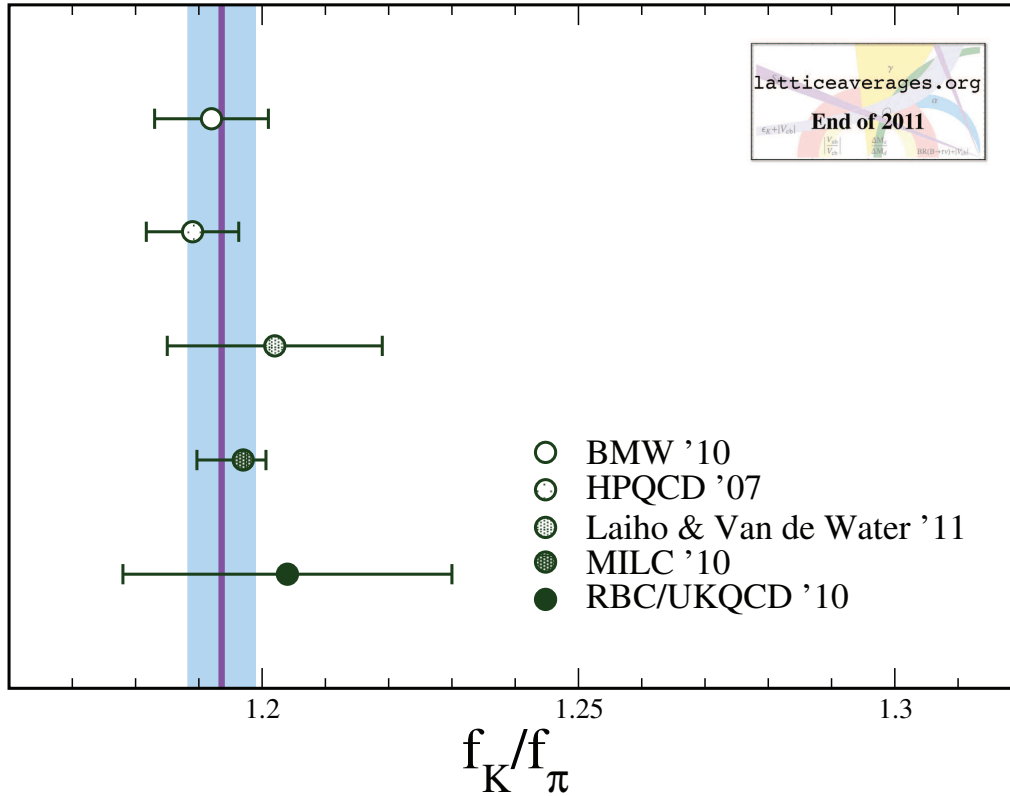


# Light quark mass ratio



Errors inflated by  $\sim 1.4$  due to somewhat low confidence level. Still  $\sim 10\sigma$  from zero.

# $f_K/f_\pi$



$$\frac{\Gamma(K \rightarrow \ell \bar{\nu}_\ell)}{\Gamma(\pi \rightarrow \ell \bar{\nu}_\ell)} = \left( \frac{|V_{us}|}{|V_{ud}|} \right)^2 \left( \frac{f_K}{f_\pi} \right)^2 \frac{m_K \left( 1 - \frac{m_\ell^2}{m_K^2} \right)^2}{m_\pi \left( 1 - \frac{m_\ell^2}{m_\pi^2} \right)^2} \left[ 1 + \frac{\alpha}{\pi} (C_K - C_\pi) \right] \quad (5)$$

# New results for $f_K/f_\pi$

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New results from the HISQ (Highly Improved Staggered Quark) action, with lattices generated by the MILC collaboration:

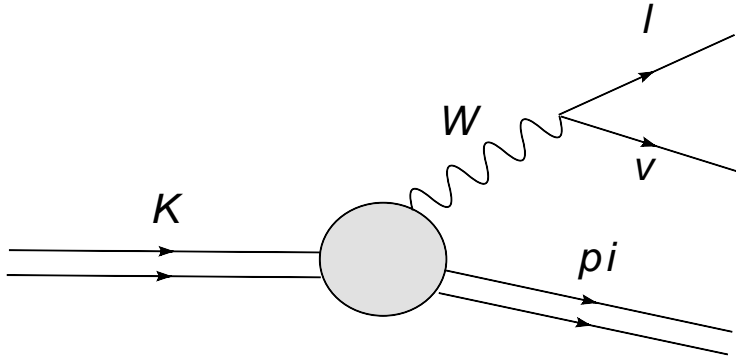
- Physical light quark masses
- Smaller taste-breaking effects than the previous MILC ensembles
- Multiple lattice spacings
- 2+1+1 flavors of sea quarks

MILC: [1.1947\(45\)](#), PRL 110, 172003 (2013)

HPQCD: [1.1916\(21\)](#), arXiv:1303.1670

Previous World Average: [1.1936\(53\)](#)

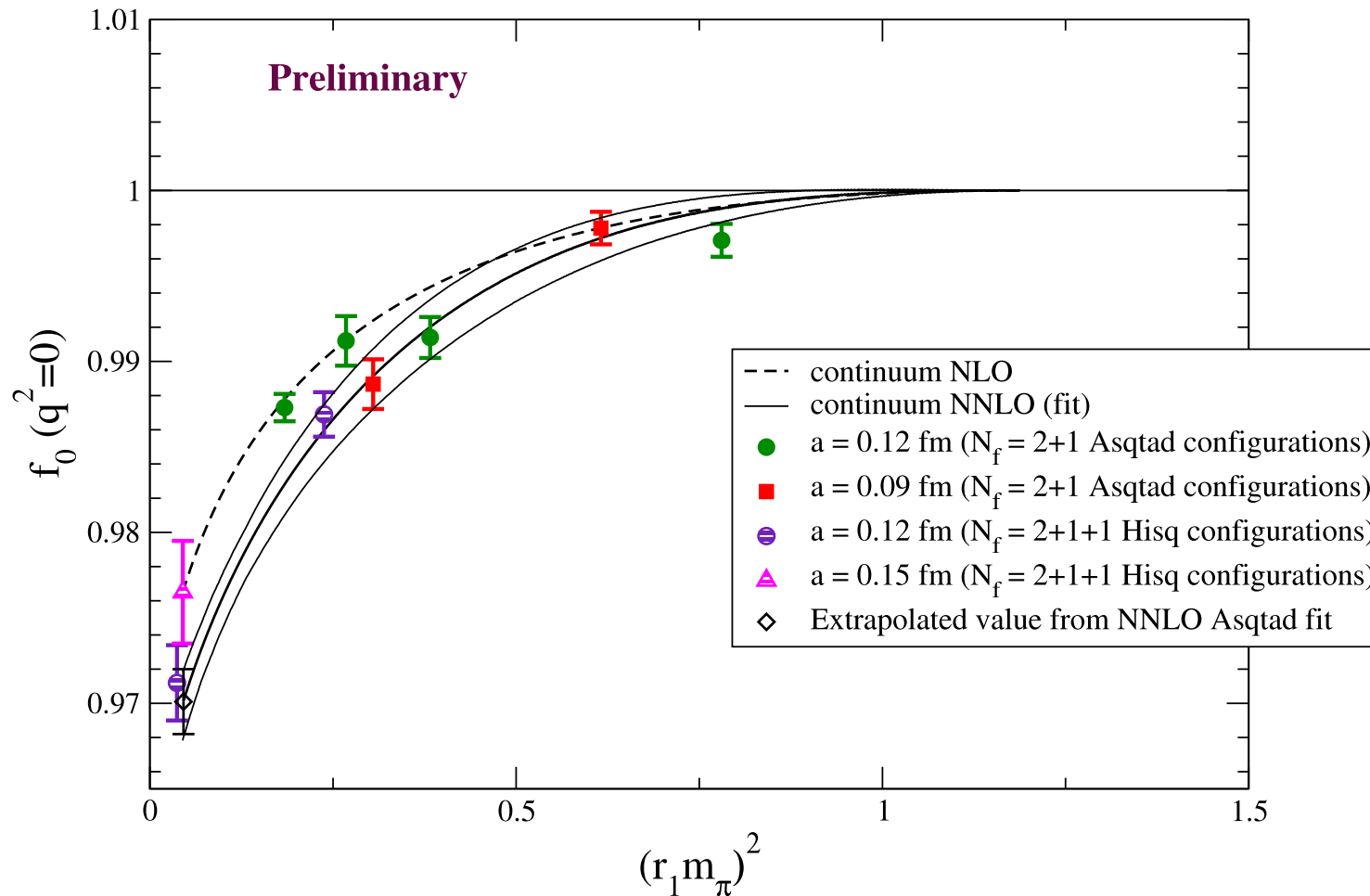
# $K \rightarrow \pi \ell \nu$



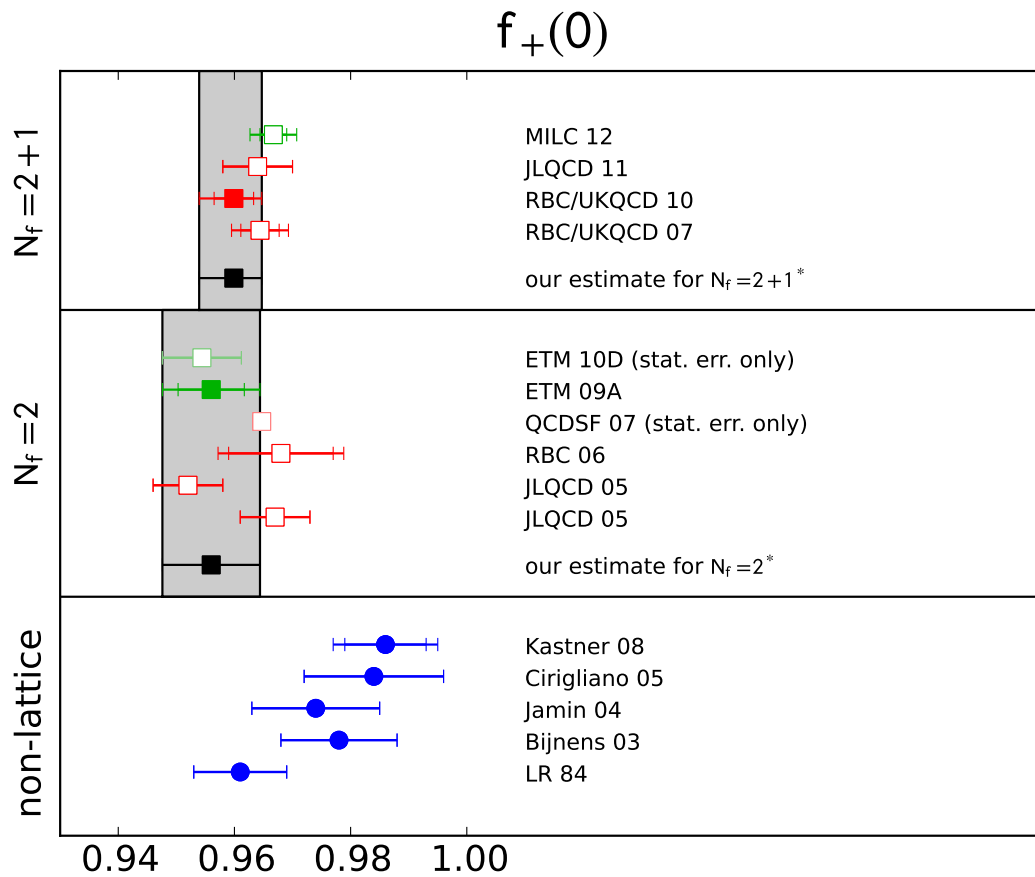
$$\Gamma_{K\ell 3} = \frac{G_F^2 m_K^5}{192\pi^3} C_K^2 S_{EW} (|V_{us}| f_+^{K^0\pi^-}(0))^2 I_{K\ell} (1 + \delta_{EM}^{K\ell} + \delta_{SU(2)}^{K\pi})^2, \quad (6)$$

where  $S_{EW} = 1.0232(3)$  is the short-distance electroweak correction,  $C_K$  is a Clebsch-Gordan coefficient,  $f_+^{K^0\pi^-}(0)$  is the form factor at zero momentum transfer, and  $I_{K\ell}$  is a phase-space integral that is sensitive to the momentum dependence of the form factors. The quantities  $\delta_{EM}^{K\ell}$  and  $\delta_{SU(2)}^{K\pi}$  are long-distance EM corrections and isospin breaking corrections, respectively.

# MILC/Fermilab $f_+(0)$



# $K \rightarrow \pi \ell \nu$

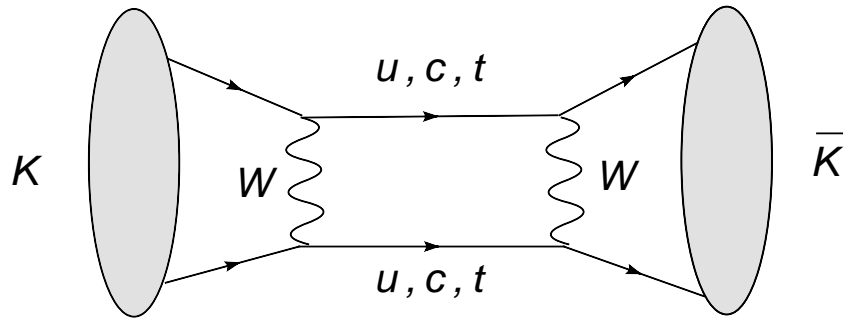


FLAG 2013 preliminary average



# $B_K$

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$$|\epsilon_K| = C_\epsilon \kappa_\epsilon B_K A^2 \bar{\eta} \left\{ -\eta_1 S_0(x_c) (1 - \lambda^2/2) + \eta_3 S_0(x_c, x_t) + \eta_2 S_0(x_t) A^2 \lambda^2 (1 - \bar{\rho}) \right\}$$

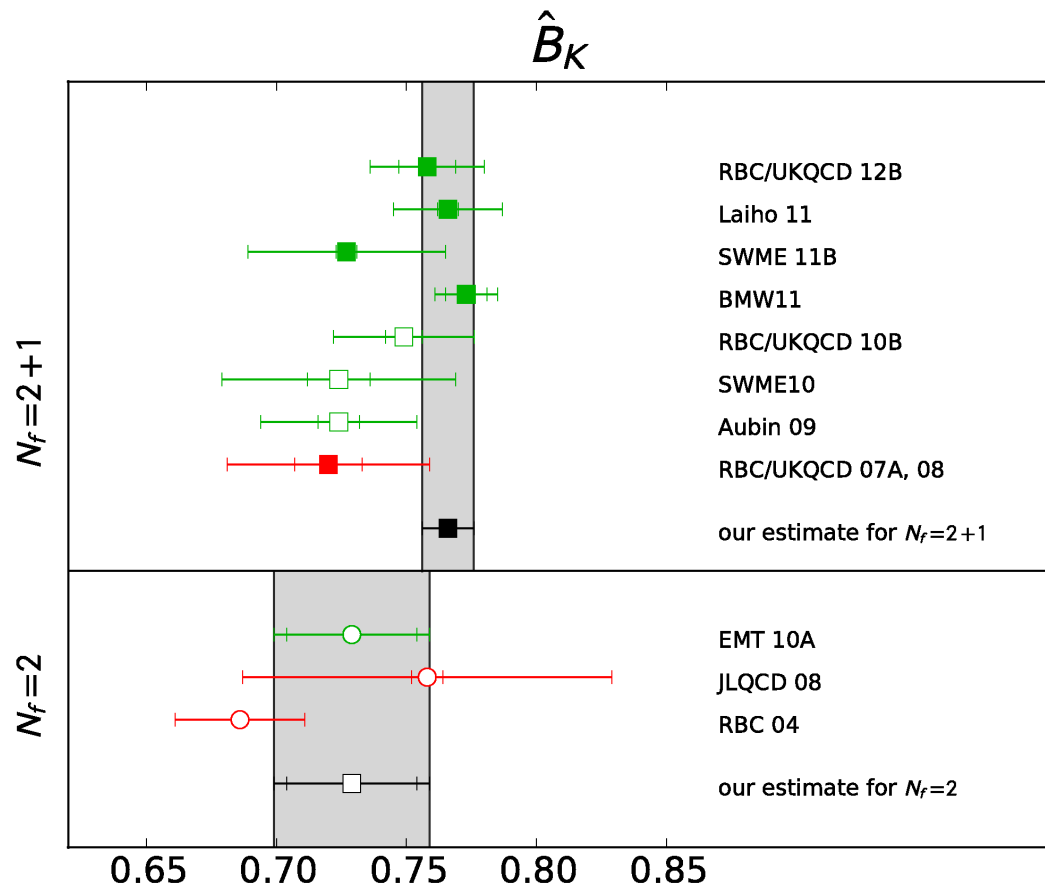
where  $C_\epsilon$  is a collection of experimentally determined parameters,  $\kappa_\epsilon$  represents long-distance corrections and a correction due to the fact that  $\phi_\epsilon \neq 45$  degrees, the  $\eta_i S_0$  are perturbative coefficients, the terms in blue are CKM matrix elements in Wolfenstein parameterization.

# $B_K$ calculations

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- RBC/UKQCD: Domain wall fermions, new non-perturbative renormalization scheme (non-exceptional momenta) with smaller systematic errors. 3 lattice spacings (1 with different action).
- JL and Van de Water: Mixed action domain wall on staggered. Adopted new renormalization scheme of RBC/UKQCD. 3 lattice spacings.
- SBW: improved staggered action on the MILC (staggered) ensembles and 4 lattice spacings. Perturbative matching.
- BMW: Clover fermion action, so explicit chiral symmetry breaking. Wrong-chirality operators are small (1%) due to smearing of the action. Non-perturbative matching to perturbation theory performed at high scale to minimize matching error. Physical light quark masses and large volumes. 4 lattice spacings.

# $B_K$



# $K \rightarrow \pi\pi$

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$$\text{Re}(\varepsilon'_K/\varepsilon) \sim \frac{\omega}{\sqrt{2}|\varepsilon_K|} \left( \frac{\text{Im}(A_2)}{\text{Re}(A_2)} - \frac{\text{Im}(A_0)}{\text{Re}(A_0)} \right) \quad (7)$$

where  $\omega = \text{Re}(A_2)/\text{Re}(A_0)$ ,  $A(K^0 \rightarrow \pi\pi(I)) = A_I e^{i\delta_I}$ .

$\text{Re}(A_2)$  is well known experimentally, and serves as a benchmark.

$\varepsilon'/\varepsilon$  requires the  $\Delta I = 1/2$  channel. This could provide a very important constraint on new physics with reduced hadronic uncertainties. How well does lattice have to do in the  $\Delta I = 1/2$  channel to be interesting to phenomenology?

$$\text{Re}(\varepsilon'_K/\varepsilon_K) = (1.68 \pm 0.19) \times 10^{-3} \quad (8)$$

Experiment is known to  $\sim 10\%$ . Even a 30% theory error would be interesting.

# $K \rightarrow \pi\pi$ matrix elements on the lattice

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Maiani-Testa no-go theorem tells us that we cannot extract physical matrix elements from Euclidean correlation functions with multi-hadron states.

Difficulties simulating at physical kinematics for  $K \rightarrow \pi\pi$  matrix elements avoided by using Lellouch-Lüscher finite volume method. This is still costly. Most straightforward implementation requires a large (6 fm) box, momentum insertion, and physical light quark masses.

# RBC/UKQCD

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Direct approach of Lellouch-Lüscher.

Calculation on  $32^3 \times 64 \times 32$  (DSDR) domain wall fermion ensembles, with  $a^{-1} = 1.4$  GeV and 4.5 fm box.

To give the pions momentum without having to fit excited states, twisted boundary conditions are used (Kim and Christ, Lattice 2002 [hep-lat/0210003], Sachrajda and Villadoro hep-lat/0411033).

# RBC/UKQCD $\text{Re}(A_2)$ and $\text{Im}(A_2)$

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uncertainty	$\text{Re}(A_2)$	$\text{Im}(A_2)$
statistics	4.3%	7.5%
finite lattice spacing	15%	15%
finite volume errors	6.2%	6.8%
Partial quenching effect	3.5%	1.7%
operator renormalization	1.7%	4.7%
unphysical kinematics	3.0%	0.22%
derivative of the phase shift	0.32%	0.32%
Wilson coefficient	7.1%	8.1%
total	19%	20%

Results presented in PRL 108 (2012) 141601.

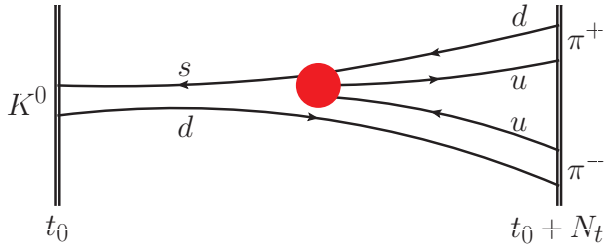
# Complications of $\Delta I = 1/2$ channel

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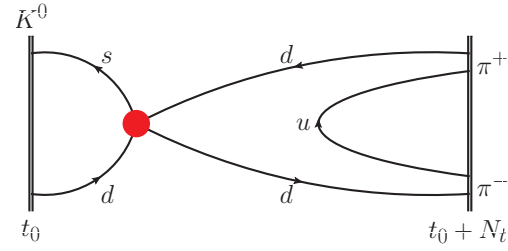
- 1) Power divergences. These can be handled by a vacuum subtraction, as shown by RBC Collaboration in the quenched approximation. Important to have chiral (expensive) quark discretization.
- 2) Enhanced finite volume effects. Can be controlled by using the unitary points (i.e. no quenching or partial quenching).
- 3) Disconnected graph. Requires brute force computing. Contributes at NLO in the  $SU(3)$  chiral expansion, so nominally sub-leading.



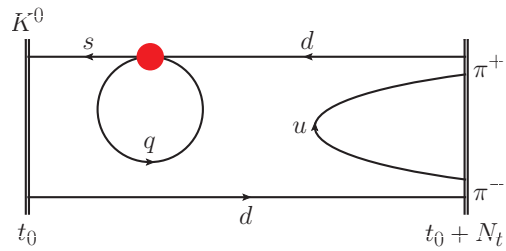
# Diagrams for $\Delta I = 1/2$ channel



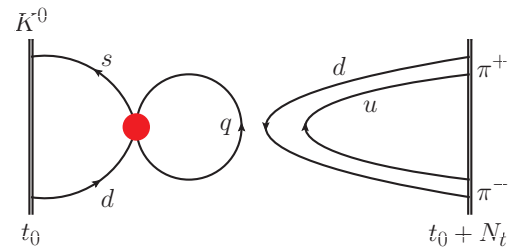
(a)



(b)



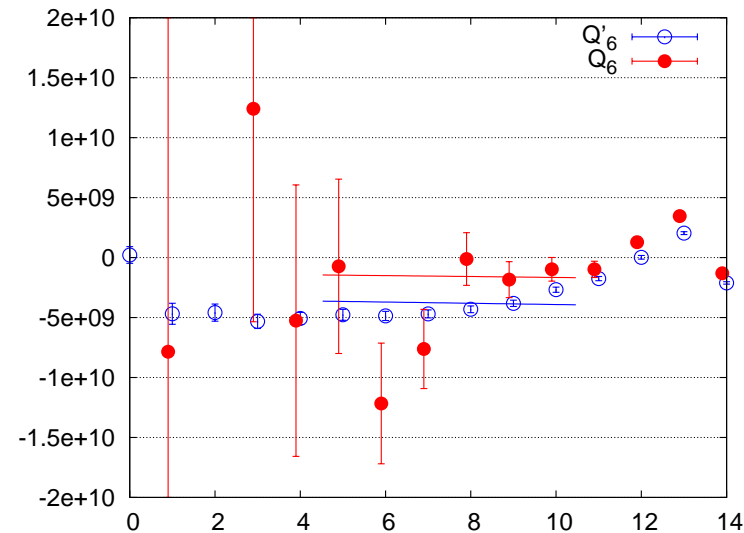
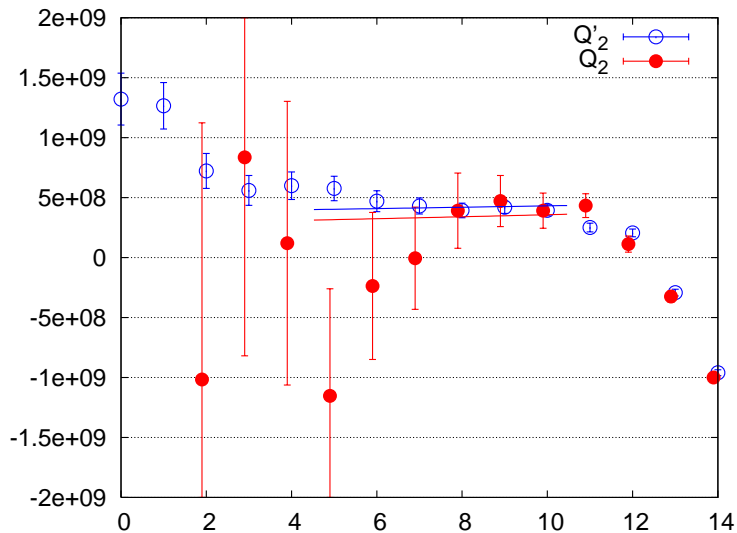
(c)



(d)

# Signals for $K \rightarrow \pi\pi$

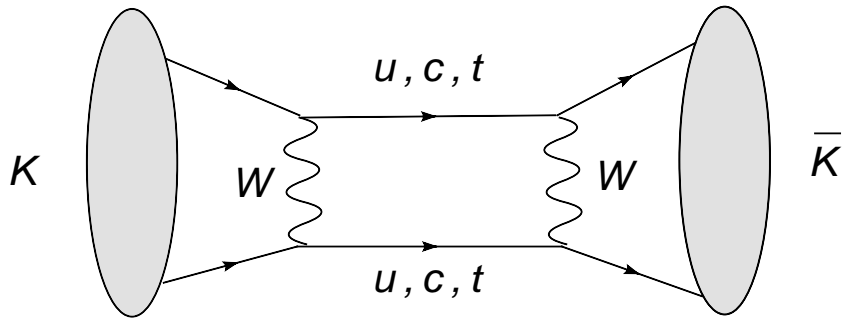
Results from RBC/UKQCD, 2010



Signals for  $K \rightarrow \pi\pi$  matrix elements at zero momentum for  $Q_2$  [relevant for  $\text{Re}(A_0)$ ] and  $Q_6$  [relevant for  $\text{Im}(A_0)$ ]. Filled symbols include disconnected diagrams and open symbols do not. Propagators were inverted on each time slice ( $T = 32$ ) for 400 configurations.

Improvements have been made [RBC/UKQCD, PRL 110 152001 (2013)], and a signal can now be resolved for  $\text{Im}(A_0)$ . Still at unphysical kinematics, but additional improvements and a new, bigger machine will help.

# $\Delta M_K$



If  $W$ 's are contracted to points leaving the internal  $u, c$  quark lines soft, this leads to long-distance corrections. Small (few %) in  $\epsilon_K$  but larger (10-30% ?) in  $\Delta M_K$ . Requires double insertion of 4-quark operators on the lattice. Method and prototype calculation given by RBC/UKQCD in arXiv:1212.5931.

- Extension of Lellouch-Lüscher method to second order in weak interactions.
- Proof of principle extraction of physical matrix elements from lattice correlation functions.
- Disconnected diagrams omitted, but could be included with enough computing.

# Conclusions

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Simple quantities from many different groups using different methods can be calculated with controlled systematic errors.

Many quantities are now precision calculations, and results are in good agreement. Prospects for improvement are excellent!

More difficult quantities like  $K \rightarrow \pi\pi$  and  $\Delta M_K$  are becoming possible. Many new things (including long-distance effects in various kaon decays) to do!

# Backup Slides

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# $K \rightarrow \pi\pi, \Delta I = 3/2, (27, 1)$

