# Nonleptonic Kaon Decays from Lattice QCD

Kaon13

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RBC and UKQCD Collaboration

# Outline

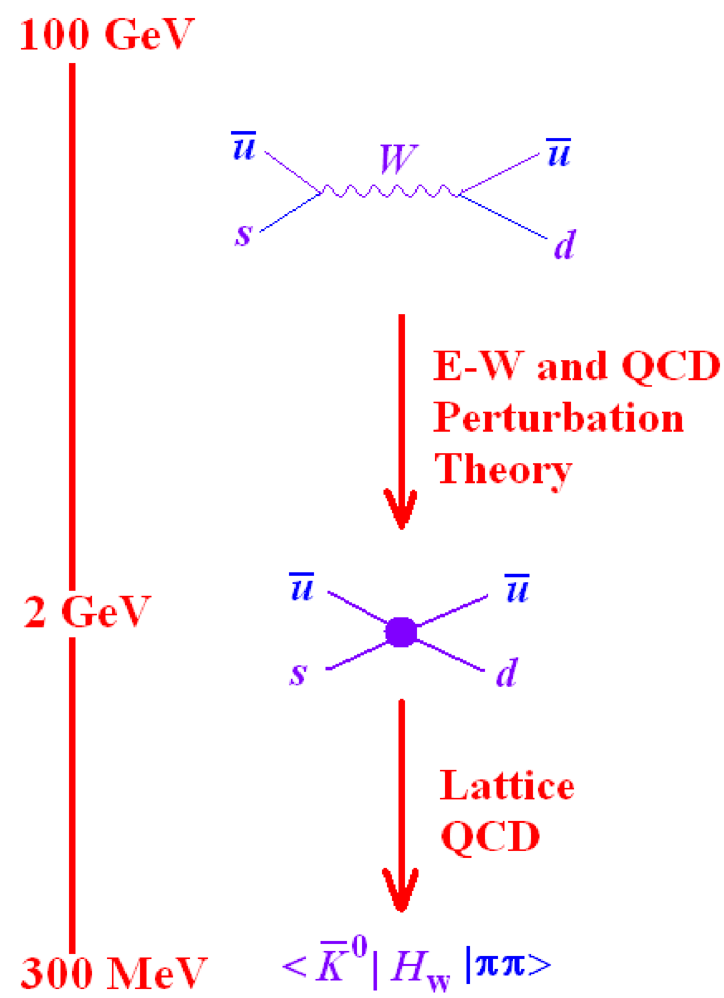
- Overview of lattice methods
- $K \rightarrow (\pi\,\pi)_{I=2} \qquad \Delta I = 3/2$
  - Strategy
  - Physical results
- $K \rightarrow (\pi\,\pi)_{I=0} \qquad \Delta I = 1/2$
  - Strategy
  - $\Delta I = 1/2$ rule
  - Prospects

# Low Energy Effective Theory

- Represent weak interactions by local four-quark Lagrangian

$$\mathcal{H}^{(\Delta S=1)} = \frac{G_F}{\sqrt{2}} V_{ud} V_{us}^* \left\{ \sum_{i=1}^{10} \left[ z_i(\mu) - \frac{V_{td}}{V_{ud}} \frac{V_{ts}^*}{V_{us}^*} y_i(\mu) \right] Q_i \right\}$$

- $V_{qq'}$ – CKM matrix elements
- $z_i$ and $y_i$ – Wilson Coefficients
- $Q_i$ – four-quark operators
- Fruit of heroic theoretical effort over the past 40 years



# Four quark operators

- **Current-current operators**

$$Q_1 \equiv (\bar{s}_\alpha d_\alpha)_{V-A}(\bar{u}_\beta u_\beta)_{V-A}$$
$$Q_2 \equiv (\bar{s}_\alpha d_\beta)_{V-A}(\bar{u}_\beta u_\alpha)_{V-A}$$

- **QCD Penguins**

$$Q_3 \equiv (\bar{s}_\alpha d_\alpha)_{V-A} \sum_{q=u,d,s} (\bar{q}_\beta q_\beta)_{V-A}$$
$$Q_4 \equiv (\bar{s}_\alpha d_\beta)_{V-A} \sum_{q=u,d,s} (\bar{q}_\beta q_\alpha)_{V-A}$$
$$Q_5 \equiv (\bar{s}_\alpha d_\alpha)_{V-A} \sum_{q=u,d,s} (\bar{q}_\beta q_\beta)_{V+A}$$
$$Q_6 \equiv (\bar{s}_\alpha d_\beta)_{V-A} \sum_{q=u,d,s} (\bar{q}_\beta q_\alpha)_{V+A}$$

- **Electro-Weak Penguins**

$$Q_7 \equiv \frac{3}{2}(\bar{s}_\alpha d_\alpha)_{V-A} \sum_{q=u,d,s} e_q(\bar{q}_\beta q_\beta)_{V+A}$$
$$Q_8 \equiv \frac{3}{2}(\bar{s}_\alpha d_\beta)_{V-A} \sum_{q=u,d,s} e_q(\bar{q}_\beta q_\alpha)_{V+A}$$
$$Q_9 \equiv \frac{3}{2}(\bar{s}_\alpha d_\alpha)_{V-A} \sum_{q=u,d,s} e_q(\bar{q}_\beta q_\beta)_{V-A}$$
$$Q_{10} \equiv \frac{3}{2}(\bar{s}_\alpha d_\beta)_{V-A} \sum_{q=u,d,s} e_q(\bar{q}_\beta q_\alpha)_{V-A}$$

# Lattice Aspects

# UKQCD Collaboration

- Edinburgh
  - Rudy Arthur (CP3 Origin)
  - Peter Boyle
  - Julien Frison
  - Nicolas Garron (Trinty)
  - Jamie Hudspith
  - Karthee Sivalingam
- Southampton
  - Elaine Goode
  - Tadeusz Janowski
  - Andreas Juttner
  - Andrew Lytle
  - Marina Marinkovic
  - Antonin Portelli
  - Chris Sachrajda

# RBC Collaboration

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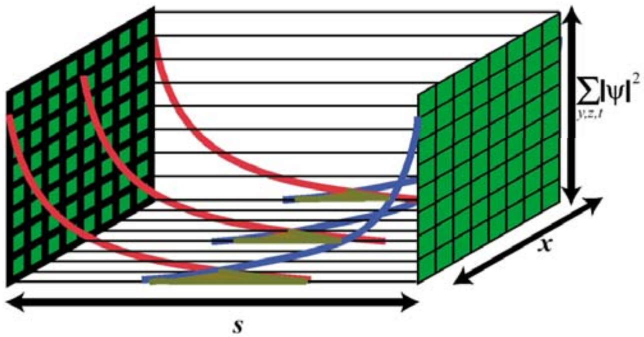
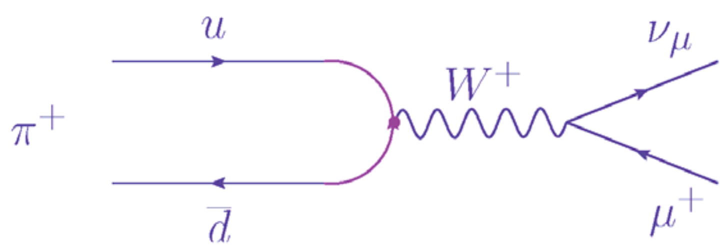
  - Matthew Lightman (St. Louis)
  - Jasper Lin
  - Meifeng Lin (Yale → ANL/BU)
  - Qi Liu (Two Sigma)
  - Robert Mawhinney
  - Greg McGlynn
  - Hantao Yin
  - Jianglei Yu
  - Daiqian Zhang

# Advances in lattice QCD



- Doubling problem solved: Use of chiral quarks now standard (domain wall fermions).
- Precise methods for normalizing lattice operators.
- Physical $\pi\,\pi$ states can be studied.
- Breathtaking advances in algorithms and computer hardware:
  - $m_\pi$=135 MeV now possible: No extrapolations in $m_q$**!**
  - $L$ = 5 – 6 fm now possible, even with chiral fermions.
- $K \rightarrow \pi\,\pi$, $\Delta M_K$, and rare $K$ decays within reach.

# State-of-the-art example: $f_\pi$



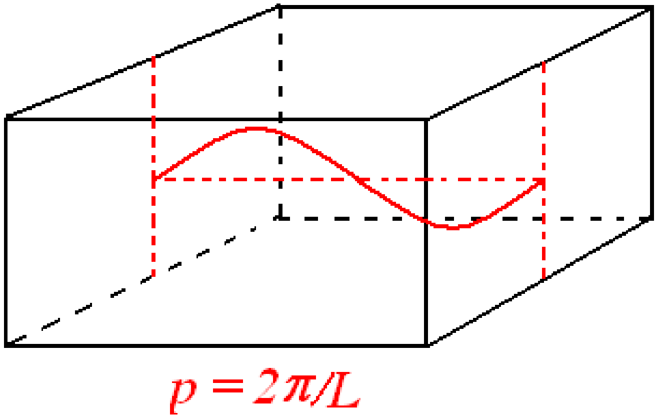

$$\langle 0|\bar{d}\gamma^5\gamma^\mu u|\pi^+(\vec{p})\rangle = f_\pi \frac{p^\mu}{\sqrt{4E_\pi(\vec{p})}}$$

$$f_\pi = N\sum_{\vec{r}} \frac{\langle A^0(\vec{r},t)O_\pi(t=0)\rangle}{\langle O_\pi^\dagger(t)O_\pi(t=0)\rangle^{\frac{1}{2}}} e^{m_\pi t/2}$$

- $48^3$ x 96, 1/$a$=1.73 GeV, $m_\pi$=135 MeV, L=5.4 fm
  - $f_\pi$ = 133.0(4)$_{\rm stat}$ MeV (12 configs. – no corrections)
  - $f_\pi$ = 130.4(0.04)(0.2) MeV (Experiment)
- $64^3$ x 128, 1/$a$=2.28 GeV calculations underway

# Physical $\pi\pi$ states – Lellouch-Luscher

- Euclidean $e^{-Ht}$ projects onto $|\pi\pi(\vec{p}=0)\rangle$
- Use finite-volume quantization.
- Adjust volume so 1st or 2nd excited state has correct $p$.




- Correctly include $\pi$ - $\pi$ interactions, including normalization.
- Requires extracting signal from non-leading large ***t*** behavior:

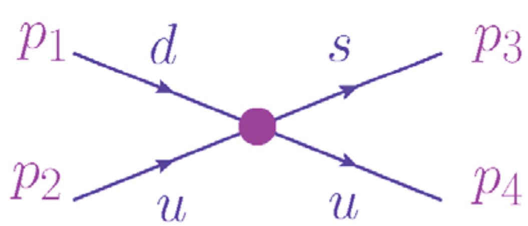
$$G(t) \sim c_0 e^{-E_0 t} + c_1 e^{-E_1 t}$$

# Operator Normalization
## (Rome-Southampton)

- Wilson expansion for $H_W$ contains four-quark operators normalized in the $\overline{\text{MS}}$ scheme.
- Use non-perturbative methods to convert lattice operators to RI scheme:
  - Evaluate Landau-gauge, off-shell Green's functions:

$$\left(\Gamma(p_1,p_2,p_3,p_4)_j\right)^{\alpha\beta\gamma\delta}_{abcd} = \prod_{i=1}^{4}\left(\int d^4x_i e^{ip_i\cdot x_i}\right)\left\langle \overline{q}^{\alpha}_a(x_1)\overline{q}^{\beta}_b(x_2)O_j q^{\delta}_c(x_3)q^{\gamma}_d(x_4)\right\rangle$$

  - Impose normalization conditions: $\mathrm{tr}\{P_i\Gamma_j\} = F_{ij}$
- Use continuum perturbation theory to convert RI to $\overline{\text{MS}}$



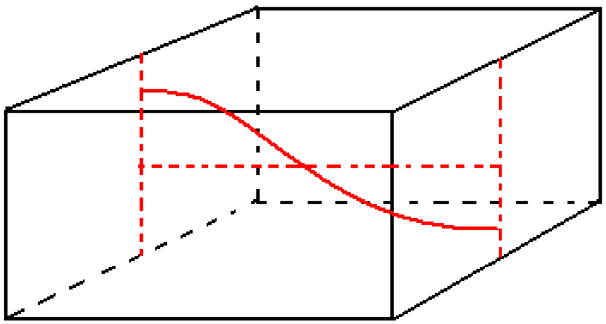
# Operator Normalization
## (Refinements)

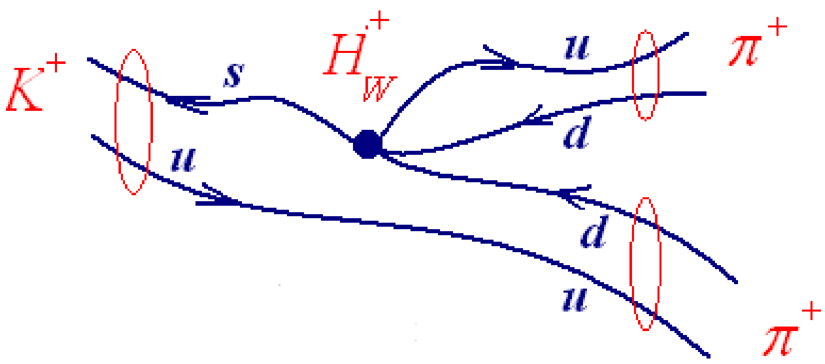
- Use chiral fermions (DWF): good short-distance chiral symmetry controls operator mixing ($L_s$=16 and 32)
- Impose normalization conditions $\mathrm{tr}\{P_i \Gamma_j\} = F_{ij}$ at infrared-safe, non-exceptional momenta, at a large, Euclidean energy scale $\mu$.
- Use a series of finer lattice ensembles to non-perturbatively run $\mu$ up to 3 GeV (or higher) before converting RI to $\overline{\mathrm{MS}}$.
- Use twisted boundary conditions to allow matching between ensembles at equal physical momenta without varying momentum direction – freeze ~~O(4)~~ $a^2$ artifacts.

$$\Delta \mathbf{I} = \mathbf{3/2}$$

# $\Delta$ I = 3/2 $K \rightarrow \pi\pi$

- Three operators contribute $O^{(27,1)}$, $O^{(8,8)}$ and $O^{(8,8)m}$.
- Use isospin to relate to $K^+ \rightarrow \pi^+\pi^+$.
- Use anti-periodic boundary conditions for *d* quark. (Changhoan Kim, hep-lat/0210003).
- **Achieve essentially physical kinematics!** (63 → 146 configurations )
  - $m_\pi$ = 142.9(1.1) MeV
  - $m_K$ = 511.3(3.9) MeV
  - $E_{\pi\pi}$ = 492(5.5) MeV

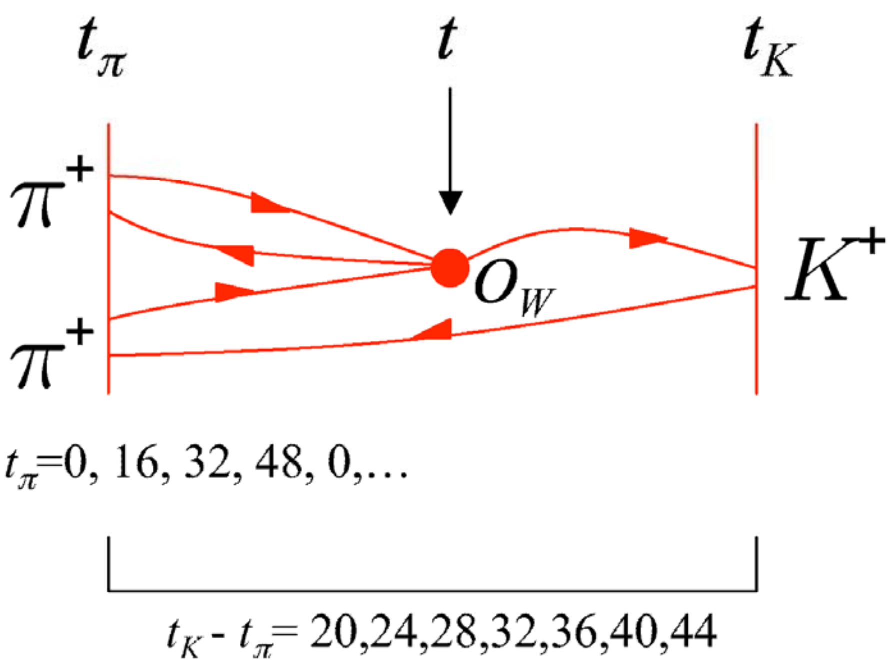


# Computational Set-up

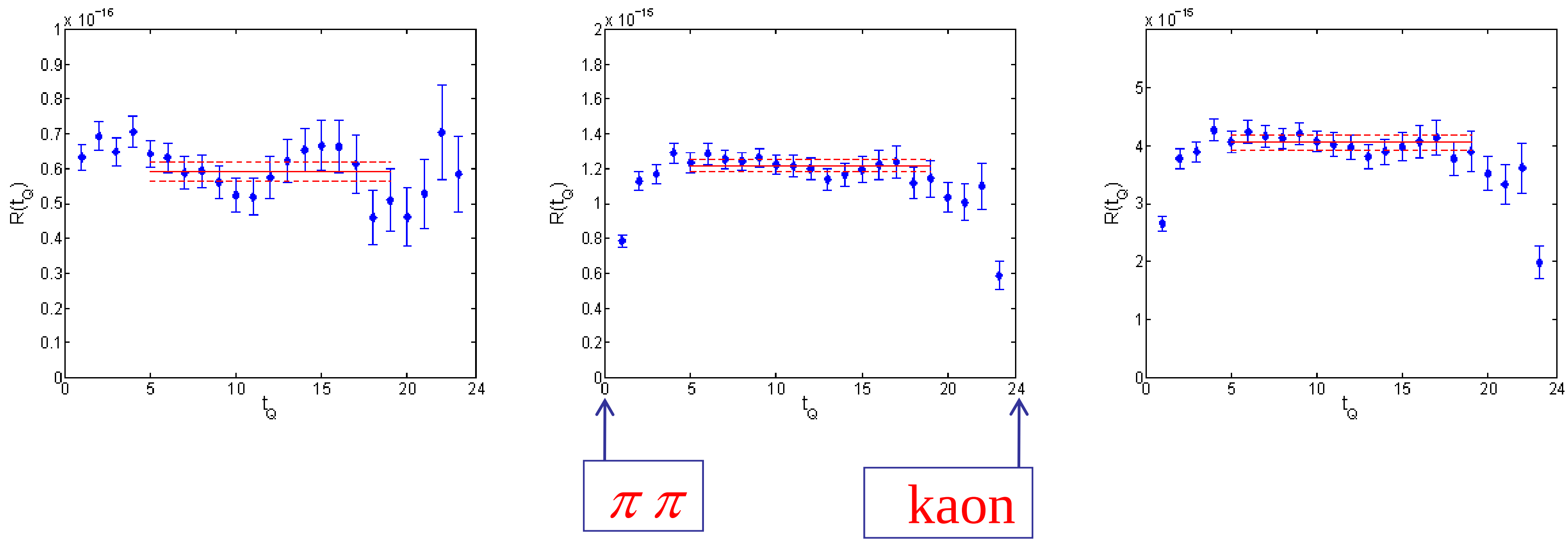
(Lightman and Goode)

- Use anti-periodic boundary conditions for $d$ quark in two directions (average over three choices).
- Fix $\pi - \pi$ source at $t = 0$, vary location of $O_W$ and kaon source.



# $<\pi\,\pi\,|\,O\,|\,K>$ from 146 configurations

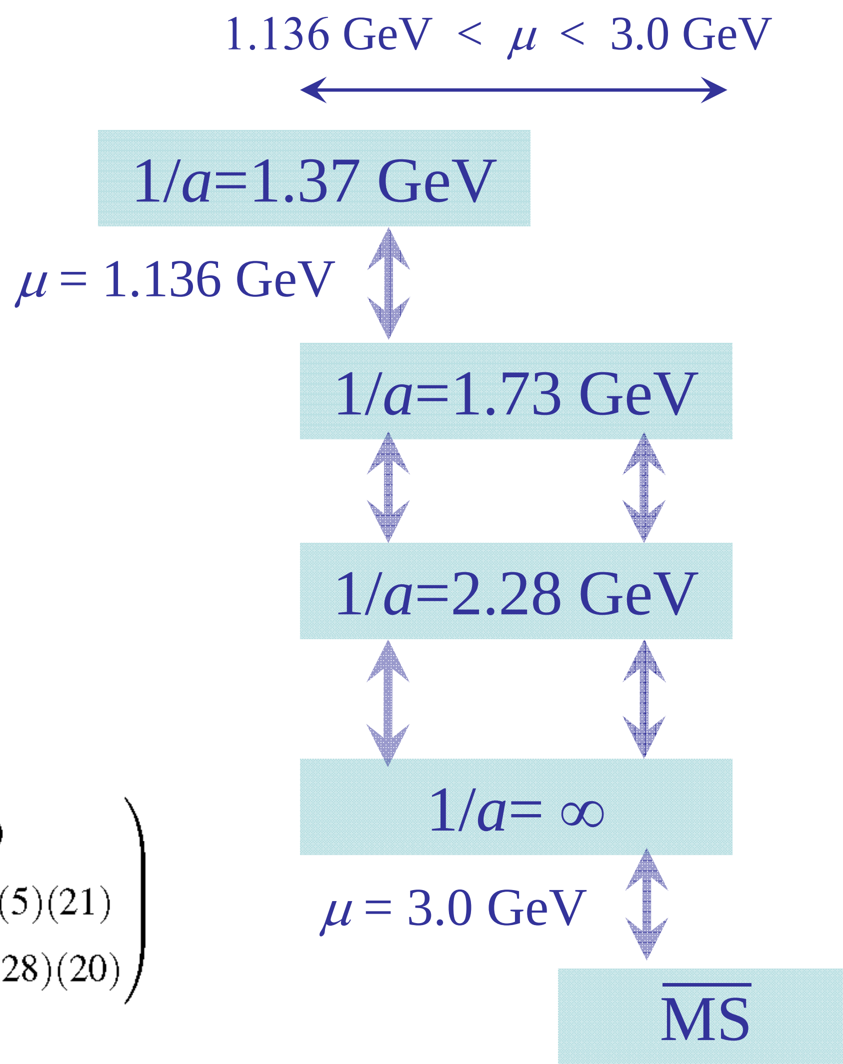
$O^{(27,1)}$ $O^{(8,8)}$ $O^{(8,8)m}$




Plot ratio of correlators: $$\frac{C^i_{K\pi\pi}(t)}{C_K(t_K - t)C_{\pi\pi}(t)} = \frac{\mathcal{M}_i}{Z_K Z_{\pi\pi}}$$

# Relate lattice and continuum operators

- Calculation is performed on $1/a$=1.37 GeV lattice.
- Matching to perturbative $\overline{\mathrm{MS}}$ scheme is unreliable at scale $\mu \sim 1/a$ **!**
- Carry out sequence of NP RI matching steps:




$$Z^{\overline{\mathrm{MS}},(\mathrm{latt})}_{(\not{q},\not{q})}(\mu) = \begin{pmatrix} 0.424(4)(4) & 0 & 0 \\ 0 & 0.472(6)(8) & -0.020(5)(21) \\ 0 & -0.067(23)(30) & 0.572(28)(20) \end{pmatrix}$$

# Determine physical $A_2$

- Recall $\langle \pi\pi(I=2)|\mathcal{L}_W(0)|K\rangle = A_2 e^{i\delta_2}$

$$A_2 = \frac{\sqrt{3}}{2\sqrt{2}} \frac{1}{\pi q_\pi} \sqrt{\frac{\partial\phi}{\partial q_\pi} + \frac{\partial\delta}{\partial q_\pi}} L^{3/2} a^{-3} G_F V_{ud} V_{us} \sqrt{m_K} E_{\pi\pi} \times \sum_{i,j} C_i(\mu) Z_{ij}(\mu) \langle \pi\pi|Q_j|K\rangle$$

- Re($A_2$) dominated by single operator $O^{(27,1)}$.
- Im($A_2$) dominated by the operators $O^{(8,8)}$ and $O^{(8,8)m}$
- Determine Lellouch-Luscher factor.

$$\frac{\partial\phi}{\partial q_\pi} = 5.038(34) \quad \frac{\partial\delta}{\partial q_\pi} = -0.2954(20)$$

# Determine physical $A_2$

- $\mathrm{Re}(A_2) = (1.436 \pm 0.063_{stat} \pm 0.258_{sys})\ 10^{-8}$ GeV

Experiment: 1.479(4) $10^{-8}$ GeV

- $\mathrm{Im}(A_2) = -(6.29 \pm 0.46_{stat} \pm 1.20_{sys})\ 10^{-13}$ GeV

- Error estimates:

| | $\mathrm{Re}A_2$ | $\mathrm{Im}A_2$ |
|---|---|---|
| lattice artefacts | 15% | 15% |
| finite-volume corrections | 6.2% | 6.8% |
| partial quenching | 3.5% | 1.7% |
| renormalization | 1.8% | 5.6% |
| unphysical kinematics | 0.4% | 0.8% |
| derivative of the phase shift | 0.97% | 0.97% |
| Wilson coefficients | 6.6% | 6.6% |
| Total | 18% | 19% |

# $\Delta$ I = 3/2 Prospects

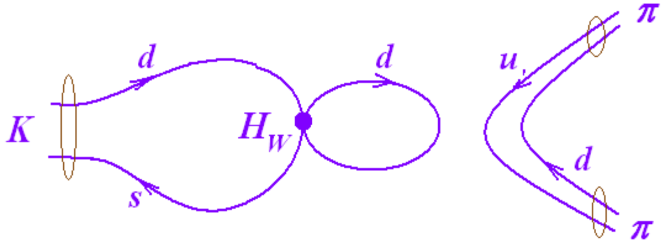
(Tadeusz Janowski and Daiqian Zhang)

- Next phase of calculation with two lattice spacings underway ($m_\pi$=135 MeV, L=5.4 fm)
  - $48^3 \times 96$, $1/a$=1.73 GeV
  - $64^3 \times 128$, $1/a$=2.28 GeV
- First $48^3 \times 96$ results (22 configs, preliminary):
  - $\mathrm{Re}(A_2) = (1.401 \pm 0.028_{stat})\ 10^{-8}$ GeV
  - $\mathrm{Im}(A_2) = -(6.29 \pm 0.46_{stat} \pm 1.20_{sys})\ 10^{-13}$ GeV
- Earlier $32^3 \times 64$ results (146 configs, published)
  - $\mathrm{Re}(A_2) = (1.436 \pm 0.063_{stat} \pm 0.258_{sys})\ 10^{-8}$ GeV
  - $\mathrm{Im}(A_2) = -(-6.54 \pm 0.13)\ 10^{-13}$ GeV
- Experiment: $\mathrm{Re}(A_2) = 1.479(4)\ 10^{-8}$ GeV

$$\Delta I = 1/2$$

# $\Delta I = 1/2$ $K \rightarrow \pi\pi$
# (Qi Liu)

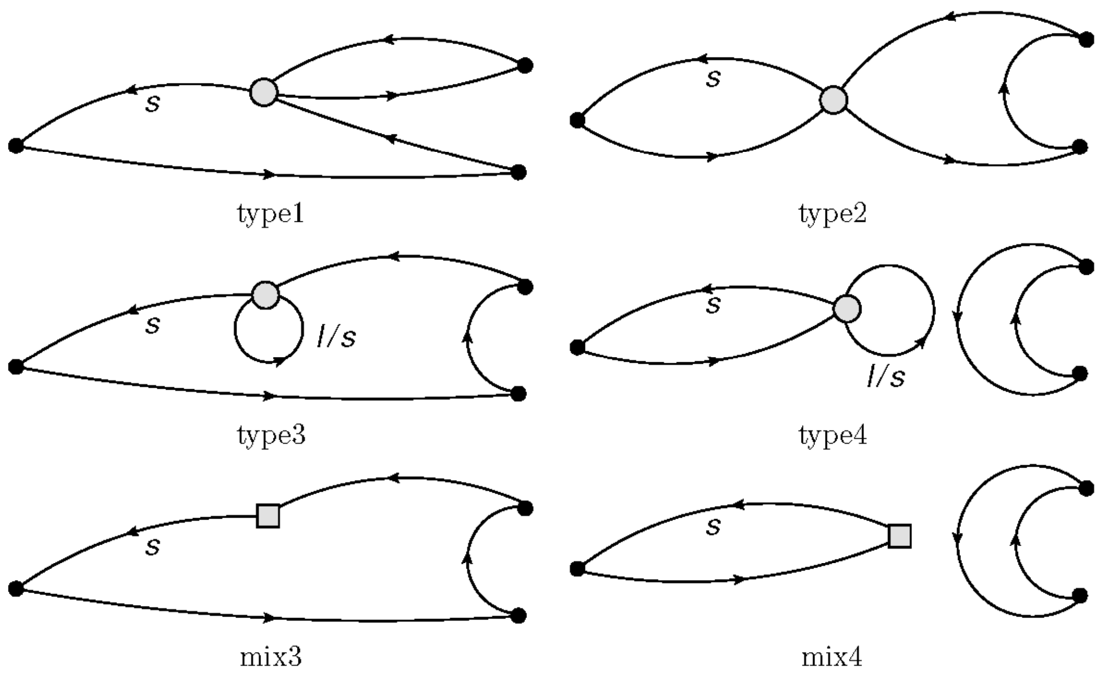
- Made much more difficult by disconnected diagrams:




- 16[3] x 32 ensemble (arXiv:1106.2714 [hep-lat])
  - $1/a$ = 1.73 GeV, $m_\pi$ = 420 MeV, $L$ = 1.8 fm
  - Use 8000 time units, measure every 10 (800 configs.)
- 24[3] x 64 ensemble (22 x harder)
  - $1/a$ = 1.73 GeV, $m_\pi$ = 329 MeV, $L$ = 2.8 fm
  - Use 5520 time units, measure every 40 (138 configs.)
- Adjust valence strange mass for on-shell, threshold kinematics ($\pi\pi$ state is unitary)

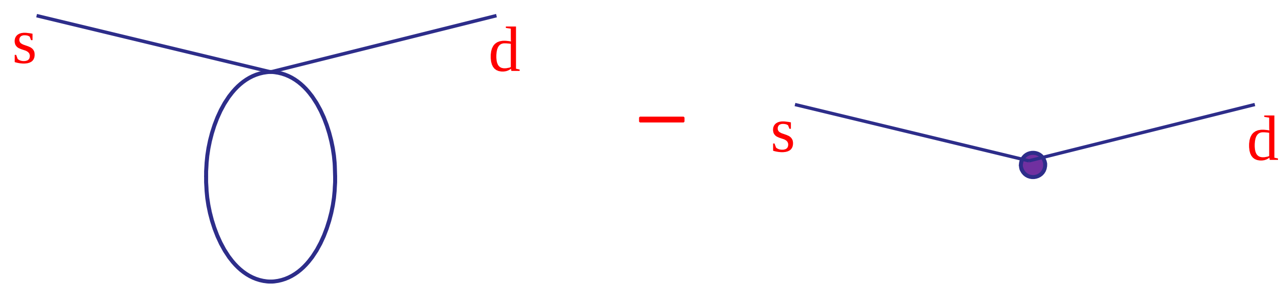
# $\Delta I = 1/2 \; K \rightarrow \pi\pi$

- Code 48 different contractions of four types:



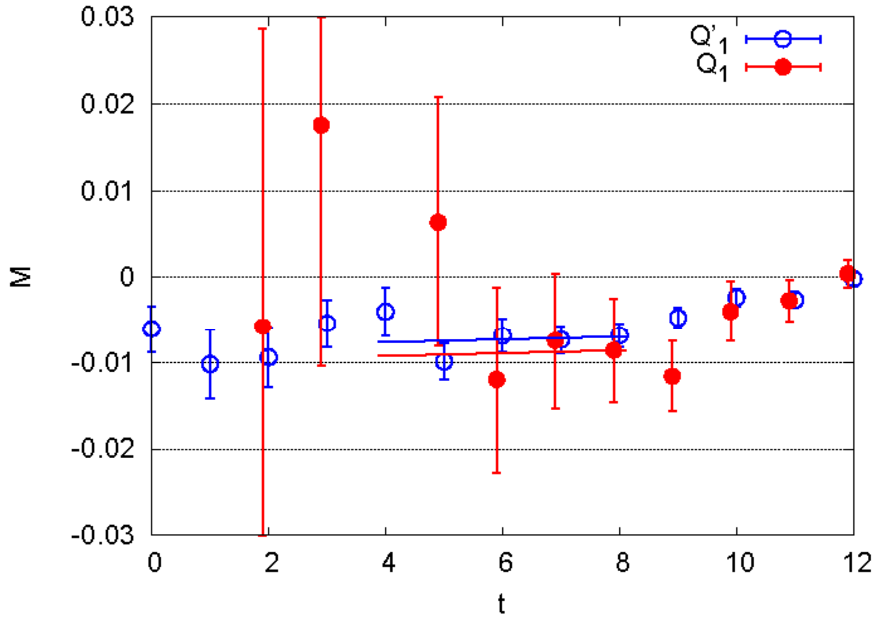
# Substantially improved methods

- Improve statistics using sources at each of 32 or 64 times
- Accelerate inversions with low-mode deflation or eigCG
- Reduce vacuum coupling by separating pion sources
- Subtract divergent $\bar{s}d$ and $\bar{s}\gamma^5 d$ terms
  - Does not affect on-shell amplitudes
  - Suppress $1/a^2$-enhanced excited state contributions.

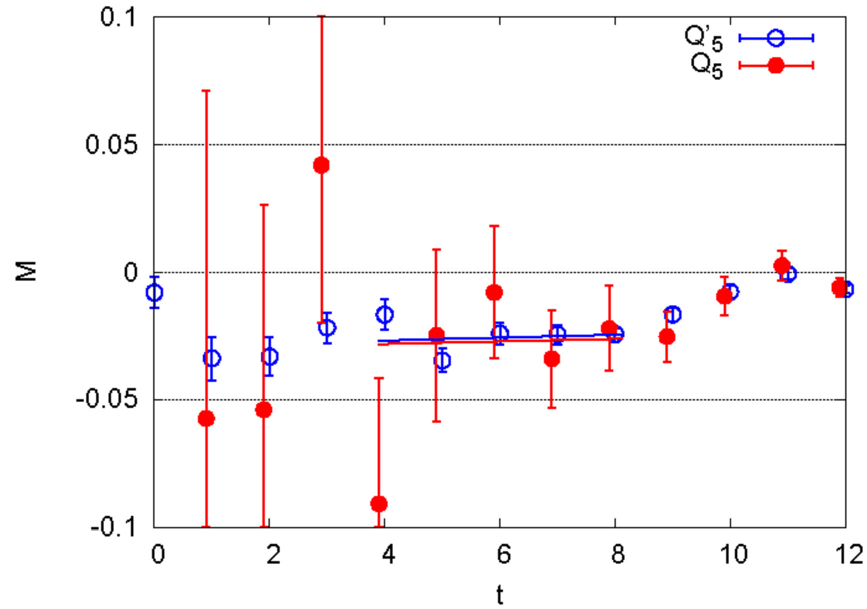


# $\Delta I = 1/2\ \ K \rightarrow \pi\pi \quad 24^3 \times 64$

Q2 - largest part of Re($A_0$)




Q6 - largest part of Im($A_0$)




$\Delta = 12\ K - \pi\pi$ separation

Full amplitude

( ′ ) Drop disconnected

| $m_\pi$(MeV) | $m_K$(MeV) | Re($A_0$) | Re($A_0'$) | Im($A_0$) | Im($A_0'$) | Re($A_2$) | Im($A_2$) |
|---|---|---|---|---|---|---|---|
| 329.3 | 662.1 | 31.1(4.5) | 27.8(0.8) | -33(15) | -36.3(16) | 2.668(14) | -0.6509(34) |

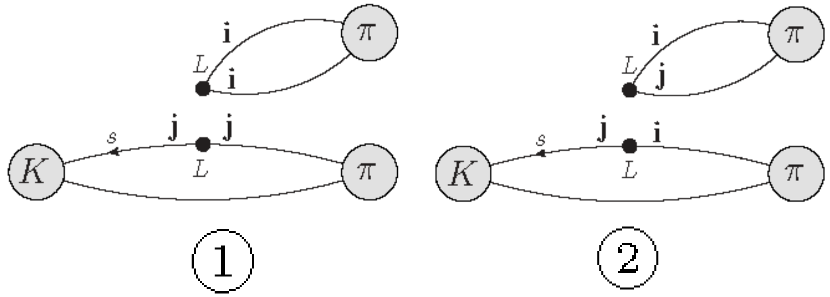
# Explain $\Delta I = 1/2$ rule
## (Q Liu)

- Understand the large ratio: $\mathrm{Re}(A_0)/\mathrm{Re}(A_2) = 22.4$
- Our results:
  - $16^3$ $m_K = 877$ MeV, $m_\pi = 422$ MeV $\quad \dfrac{\mathrm{Re}(A_0)}{\mathrm{Re}(A_2)} = 9.1(21)$
  - $24^3$ $m_K = 662$ MeV, $m_\pi = 329$ MeV $\quad \dfrac{\mathrm{Re}(A_0)}{\mathrm{Re}(A_2)} = 12.0(17)$
- 2x explained by Wilson coefficients.
- Largest part of $A_0$ and $A_2$ comes from two contractions in $<\pi\pi|Q_2|K>$
- Look at actual amplitudes.

# Explain $\Delta I$ = 1/2 rule

## (Q Liu)

- Two current-current diagrams dominate:




- Where

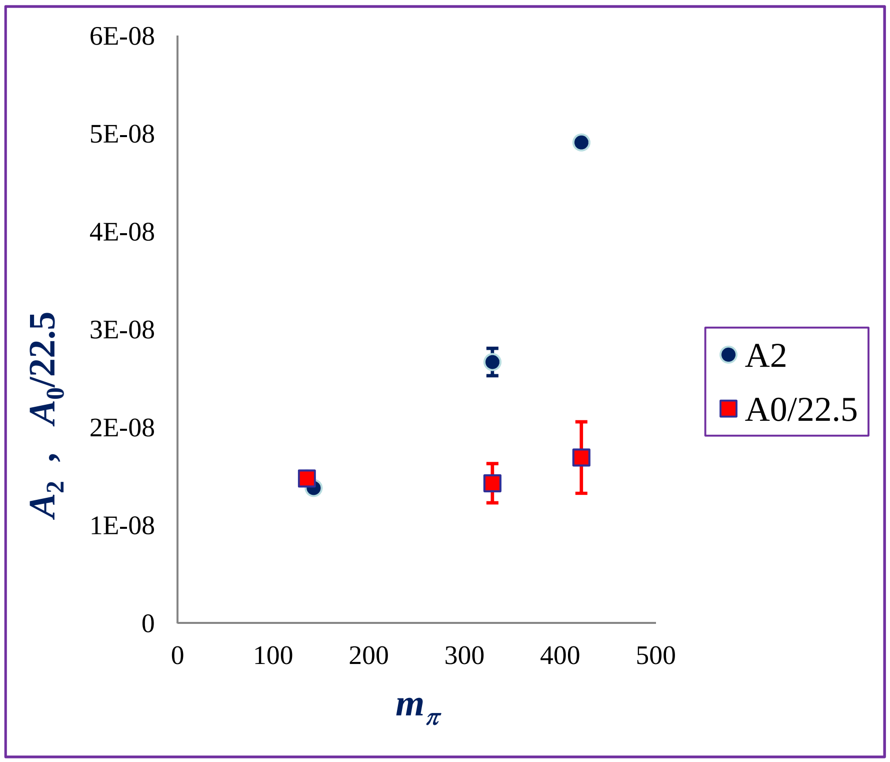
$$A_{0,2}(t_\pi, t_{\rm op}, t_K) \approx i\frac{1}{\sqrt{3}}\{2\cdot ① - ②\}$$

$$A_{2,2}(t_\pi, t_{\rm op}, t_K) = i\sqrt{\frac{2}{3}}\{① + ②\}$$

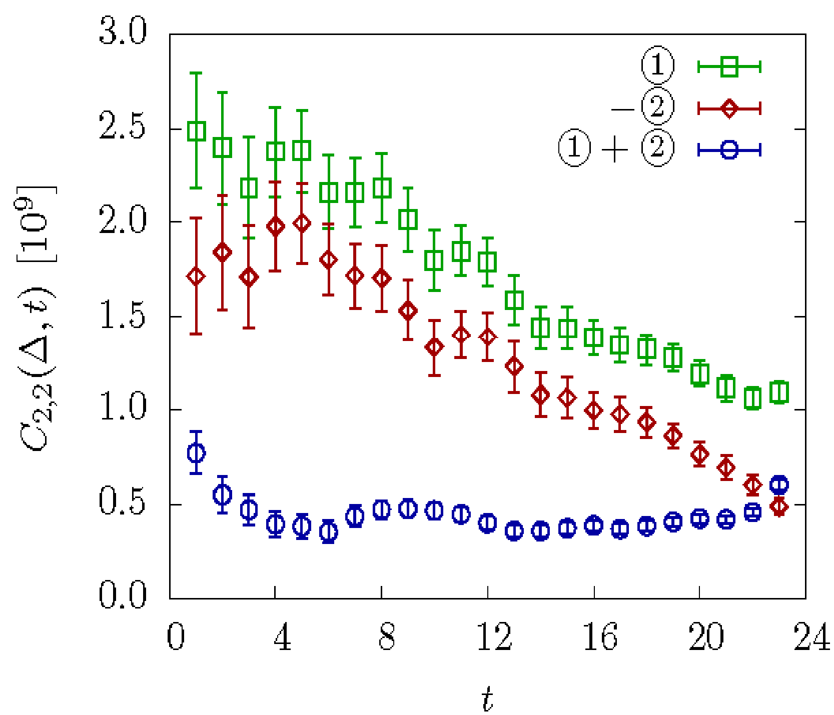
- Factorization: ② = 1/3 ①
- Actual calculation: ② = –0.7 ①

# $\Delta I = 1/2$ rule – Emerging explanation

Compare $A_2$ and $A_0/22.5$

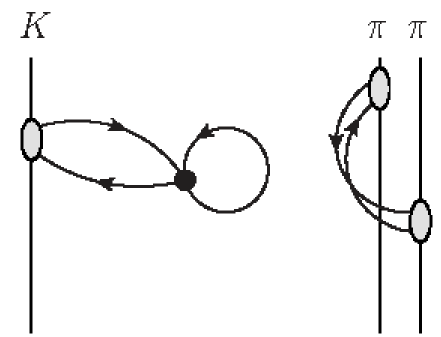


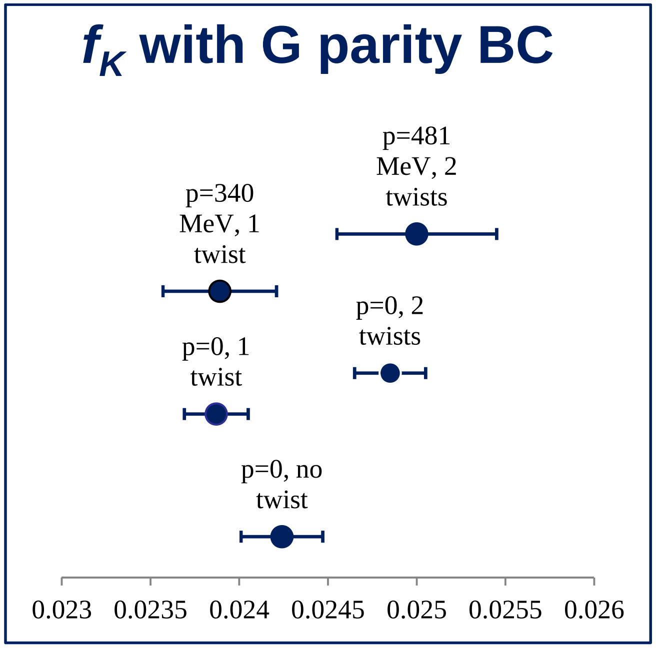

Cancellation in $A_2$



# $\Delta I = ½$ $K \rightarrow \pi\pi$ – Next steps
## (Daiqian Zhang & Chris Kelly)

- Use **all-2-all** propagators (KEK/Trinity)
  - Sum over localized sources – further suppress vacuum coupling
  - See 5x improvement in statistics for $I = 0$, $\pi$–$\pi$ scattering
- Use **G-parity** in 3 directions to achieve $p_\pi = 205$ MeV
  - Extra I = 1/2, $s'$ quark adds $e^{-m_K L}$ error.
  - G-parity code 90% written
  - Tests: $f_K$ and $B_K$ correct within errors.



# $\Delta I = 1/2$ $K \rightarrow \pi\pi$: Future

- Goal is a 20% calculation of $\varepsilon'/\varepsilon$ with all errors controlled
- Repeat $\Delta I = 3/2$ kinematics
  - Use $32^3 \times 64$ volume with $1/a = 1.37$ GeV
  - Achieve $p = 205$ MeV from **G-parity** boundary conditions
- BG/Q gives 20 x speedup
- Begin configuration generation, 7/2013
- Result expected in 2 years

# Kaon physics on the lattice
# Outlook

- Work at physical quark masses.
- DW fermions and NPR give continuum-like control of operator normalization and mixing.
- Theoretical advances allow rescattering effects to be correctly computed in Euclidean space.
- Many critical quantities can now be computed:
  - $K \rightarrow \pi\pi$ , $\Delta I$=3/2 and 1/2, $\varepsilon'/\varepsilon$
  - $m_{K_L} - m_{K_S}$
  - $K \rightarrow \pi l \bar{l}$