# Prospects for Lattice Calculations of Rare Kaon Decay Amplitudes

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# 1. Introduction

- At this conference we are seeing the very significant progress since Kaon09 in the precision of lattice results for important quantities in flavour physics in general and kaon physics in particular.

  See e.g. talks by S.Durr, J.Laiho, V.Lubicz, A.Portelli and R.Mawhinney

- It is now both necessary and possible to extend the range of physical quantities which can be studied using lattice simulations and RBC-UKQCD are leading on several important extensions in kaon physics.
  - *Standard* quantities include those in which non-perturbative QCD effects can be written as matrix elements of local composite operators:

    $$\langle 0\,|\,O(0)\,|\,h\rangle \quad \text{and} \qquad \langle h_2\,|\,O(0)\,|\,h_1\rangle\,,$$

    where $h, h_1$ and $h_2$ represent hadrons.
  - An important example is the evaluation of $K\to\pi\pi$ decay amplitudes with the goal of understanding the $\Delta I=1/2$ rule and the numerical value of $\varepsilon'/\varepsilon$. Here we have two hadrons in the final state. See the talk by N.Christ

- In this talk I will discuss the evaluation of matrix elements of non-local operators

  $$\int d^4x \int d^4y\,\langle h_2\,|\,T\{O_1(x)\,O_2(y)\}\,|\,h_1\rangle\,,$$

  and, in particular, applications to $\Delta M_K$ and rare kaon decay amplitudes.
  - Long-distance quantities are included.

## Why am I giving this talk now?

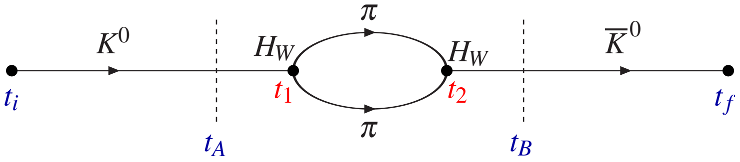


- A lattice program of computations of rare kaon decay amplitudes is challenging and we need to focus on those aspects where we can contribute effectively.
- There are people in this room who have been thinking about rare kaon decays for many years.
- We need help and advice from the wider kaon-physics community!
- At the same time it may be important for the wider community to understand what we can or cannot do, in principle and/or in practice, to guide their strategies.

The main purpose of this talk is therefore to stimulate a discussion between those interested in rare-kaon decays from different perspectives.

## *Long-distance contributions to the $K_L$ - $K_S$ mass difference*

N.H.Christ, T.Izubuchi, C.T.Sachrajda, A.Soni, J.Yu arXiv:1212.5931.




- We wish to compute the amplitude

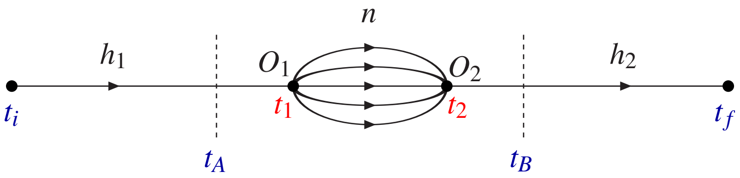
$$\mathscr{A} = \frac{1}{2}\int_{-\infty}^{\infty} dt_1\, dt_2\; T\,\langle \bar{K}^0\,|\,H_W(t_2)\,H_W(t_1)\,|\,K^0\rangle$$

  and to determine the $K_L$-$K_S$ mass difference:

$$\Delta M_K = 2\mathscr{P}\sum_\alpha \frac{\langle \bar{K}^0\,|\,H_W\,|\alpha\rangle\,\langle\alpha\,|H_W\,|\,K^0\rangle}{m_K - E_\alpha}$$

  where the sum over $|\alpha\rangle$ includes an energy-momentum integral.

- This is a significant extension of the standard calculations, where the matrix elements are of local operators.

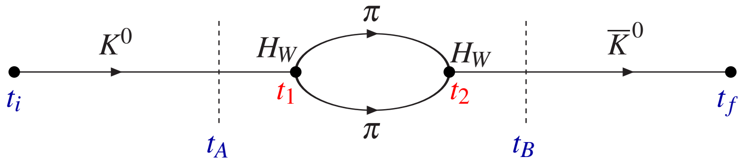



- How do you prepare the states $h_{1,2}$ in

$$\int d^4x \int d^4y \, \langle h_2 \,|\, T\{O_1(x)\, O_2(y)\} \,|\, h_1 \rangle \, ,$$

  when the time of the operators is integrated?
- The practical solution is to integrate over a large subinterval in time $t_A \le t_{x,y} \le t_B$, but to create $h_1$ and to annihilate $h_2$ well outside of this region:
- This is the natural modification of standard field theory for which the asymptotic states are prepared at $t \to \pm\infty$ and then the operators are integrated over all time.
- This approach has been successfully implemented in the $\Delta M_K$ project.

arXiv:1212.5931




- $\Delta m_K$ is given by

$$\Delta m_K \equiv m_{K_L} - m_{K_S} = \frac{1}{2m_K} 2\mathscr{P} \sum_\alpha \frac{\langle \bar{K}^0 \,|\, \mathscr{H}_W \,|\, \alpha \rangle \, \langle \alpha \,|\, \mathscr{H}_W \,|\, K^0 \rangle}{m_K - E_\alpha} = 3.483(6) \times 10^{-12}\,\text{MeV}.$$

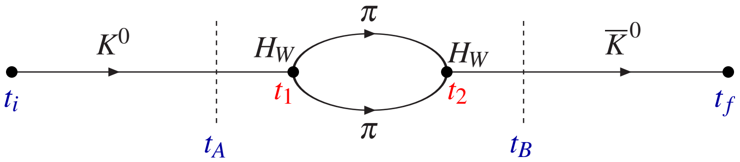
- The above correlation function gives ($T = t_B - t_A + 1$)

$$C_4(t_A, t_B; t_i, t_f) = |Z_K|^2 e^{-m_K(t_f - t_i)} \sum_n \frac{\langle \bar{K}^0 \,|\, \mathscr{H}_W \,|\, n \rangle \, \langle n \,|\, \mathscr{H}_W \,|\, K^0 \rangle}{(m_K - E_n)^2} \times \left\{ e^{(M_K - E_n)T} - (m_K - E_n)T - 1 \right\}.$$

- From the coefficient of $T$ we can therefore obtain

$$\Delta m_K^{\rm FV} \equiv 2 \sum_n \frac{\langle \bar{K}^0 \,|\, \mathscr{H}_W \,|\, n \rangle \, \langle n \,|\, \mathscr{H}_W \,|\, K^0 \rangle}{(m_K - E_n)}.$$

# Evaluating $\Delta m_K$ (cont.)




In order to evaluate $\Delta m_K$ we need to be able to:

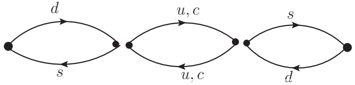
- Relate $\Delta m_K$ and $\Delta m_K^{\mathrm{FV}}$. ✓ RBC-UKQCD; N.H.Christ, G.Martinelli, CTS (in preparation)
  This is a significant extension of the theory of finite-volume effects for two-pion states: the Lüscher quantization condition, Lellouch-Lüscher factor, $\cdots$.
- Control the additional ultraviolet divergences when the weak Hamiltonians are close together. ✓ arXiv:1212.5931
  This is facilitated by the GIM mechanism which requires the presence of charm quarks.
- $\Delta S = 1$ effective weak Hamiltonian including four flavours:

$$H_W = \frac{G_F}{\sqrt{2}} \sum_{q,q'=u,c} V_{qd} V^*_{q's} (C_1 Q_1^{qq'} + C_2 Q_2^{qq'})$$

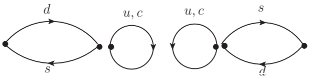
  where

$$Q_1^{qq'} = (\bar{s}_i d_i)_{V-A} (\bar{q}_j q'_j)_{V-A} \quad \text{and} \quad Q_2^{qq'} = (\bar{s}_i d_j)_{V-A} (\bar{q}_j q'_i)_{V-A}\,.$$

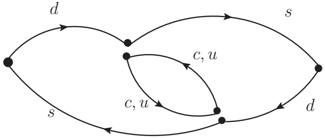
# Evaluating $\Delta m_K$ (cont.)



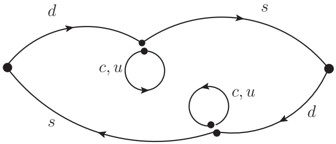

Type 1




Type 4




Type 2

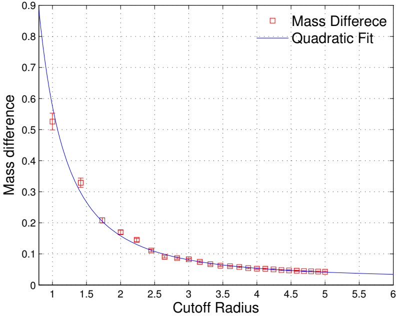



Type 3

- In our exploratory study on $16^3$ ensembles with $m_\pi = 420$ MeV, we only evaluate Type 1 and Type 2 graphs. arXiv:1212.5931
- We obtain $\Delta m_K$ in the range $\{5.81(28) - 10.58(75)\} \times 10^{-12}$ MeV as $m_K$ is varied from 563 to 839 MeV. (The physical value is $3.483(6) \times 10^{-12}$ MeV.)
- We are currently completing a full study (all diagrams) on a $24^3$ lattice with $m_\pi \simeq 330$ MeV. Jianglei Yu et al.

## Evaluating $\Delta m_K$ (cont.)

- As an example of our investigations consider the behaviour of the integrated $Q_1 - Q_1$ correlation function without GIM subtraction but with an artificial cut-off, $R = \sqrt{\{(t_2 - t_1)^2 + (\vec{x}_2 - \vec{x}_1)^2\}}$ on the coordinates of the two $Q_1$ insertions.




- The plot exhibits the quadratic divergence as the two operators come together.
- The quadratic divergence is cancelled by the GIM mechanism.

F.Mescia, C.Smith, S.Trine hep-ph/0606081

- Rare kaon decays which are dominated by short-distance FCNC processes, $K \to \pi \nu \bar{\nu}$ in particular, provide a potentially valuable window on new physics at high-energy scales.
- The decays $K_L \to \pi^0 e^+ e^-$ and $K_L \to \pi^0 \mu^+ \mu^-$ are also considered promising because the long-distance effects are reasonably under control using ChPT.
  - They are sensitive to different combinations of short-distance FCNC effects and hence in principle provide additional discrimination to the neutrino modes.
  - A challenge for the lattice community is therefore either to calculate the long-distance effects reliably or at least to determine the Low Energy Constants of ChPT.
- We are now in a position to attempt to meet this challenge.
  - We need help from the experimental and non-lattice theory communities to focus on the key issues.

There are three main contributions to the amplitude:

1. Short distance contributions: F.Mescia, C,Smith, S.Trine hep-ph/0606081

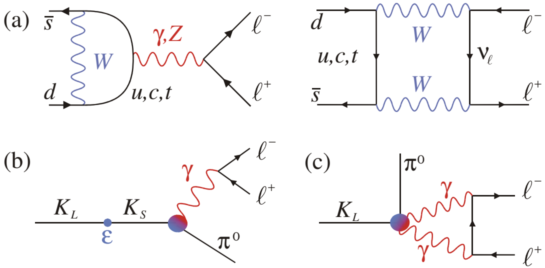
$$H_{\rm eff} = -\frac{G_F \alpha}{\sqrt{2}} V_{ts}^* V_{td} \{ y_{7V} (\bar{s}\gamma_\mu d)(\bar{\ell}\gamma^\mu \ell) + y_{7A} (\bar{s}\gamma_\mu d)(\bar{\ell}\gamma^\mu \gamma_5 \ell) \} + \text{h.c.}$$

   - Direct CP-violating contribution.
   - In BSM theories other effective interactions are possible.

2. Long-distance indirect CP-violating contribution

$$A_{ICPV}(K_L \to \pi^0 \ell^+ \ell^-) = \varepsilon A(K_1 \to \pi^0 \ell^+ \ell^-)\,.$$

3. The two-photon CP-conserving contribution $K_L \to \pi^0 (\gamma^* \gamma^* \to \ell^+ \ell^-)$.



- The current phenomenological status for the SM predictions is nicely summarised by: V.Cirigliano et al., arXiv1107.6001

$$\mathrm{Br}(K_L \to \pi^0 e^+ e^-)_{\mathrm{CPV}} = 10^{-12} \times \left\{ 15.7|a_S|^2 \pm 6.2|a_S| \left(\frac{\mathrm{Im}\,\lambda_t}{10^{-4}}\right) + 2.4 \left(\frac{\mathrm{Im}\,\lambda_t}{10^{-4}}\right)^2 \right\}$$

$$\mathrm{Br}(K_L \to \pi^0 \mu^+ \mu^-)_{\mathrm{CPV}} = 10^{-12} \times \left\{ 3.7|a_S|^2 \pm 1.6|a_S| \left(\frac{\mathrm{Im}\,\lambda_t}{10^{-4}}\right) + 1.0 \left(\frac{\mathrm{Im}\,\lambda_t}{10^{-4}}\right)^2 \right\}$$

  - $\lambda_t = V_{td} V_{ts}^*$ and $\mathrm{Im}\,\lambda_t \simeq 1.35 \times 10^{-4}$.
  - $|a_S|$, the amplitude for $K_S \to \pi^0 \ell^+ \ell^-$ at $q^2 = 0$ as defined below, is expected to be $O(1)$ but the sign of $a_S$ is unknown. $|a_S| = 1.06^{+0.26}_{-0.21}$.
  - For $\ell = e$ the two-photon contribution is negligible.
  - Taking the positive sign (?) the prediction is

$$\begin{aligned} \mathrm{Br}(K_L \to \pi^0 e^+ e^-)_{\mathrm{CPV}} &= (3.1 \pm 0.9) \times 10^{-11} \\ \mathrm{Br}(K_L \to \pi^0 \mu^+ \mu^-)_{\mathrm{CPV}} &= (1.4 \pm 0.5) \times 10^{-11} \\ \mathrm{Br}(K_L \to \pi^0 \mu^+ \mu^-)_{\mathrm{CPC}} &= (5.2 \pm 1.6) \times 10^{-12}\,. \end{aligned}$$

  - The current experimental limits (KTeV) are:

$$\mathrm{Br}(K_L \to \pi^0 e^+ e^-) < 2.8 \times 10^{-10} \quad \text{and} \quad \mathrm{Br}(K_L \to \pi^0 \mu^+ \mu^-) < 3.8 \times 10^{-10}\,.$$

G.Isidori, G.Martinelli and P.Turchetti, hep-lat/0506026

- We now turn to the CPC decays $K_S \to \pi^0 \ell^+ \ell^-$ and $K^+ \to \pi^+ \ell^+ \ell^-$ and consider

$$T_i^\mu = \int d^4x\, e^{-iq\cdot x}\, \langle \pi(p)\,|\,\mathrm{T}\{J^\mu_{\mathrm{em}}(x)\,Q_i(0)\,\}\,|\,K(k)\rangle\,,$$

  where $Q_i$ is an operator from the effective Hamiltonian.
- Gauge invariance implies that

$$T_i^\mu = \frac{\omega_i(q^2)}{(4\pi)^2}\left\{q^2(p+k)^\mu - (m_K^2 - m_\pi^2)\,q^\mu\right\}.$$

- Within ChPT the Low energy constants $a_+$ and $a_S$ are defined by

$$a = \frac{1}{\sqrt{2}}\,V_{us}^* V_{ud}\left\{C_1\omega_1(0) + C_2\omega_2(0) + \frac{2N}{\sin^2\theta_W} f_+(0) C_{7V}\right\}$$

  where $Q_{1,2}$ are the two current-current GIM subtracted operators and the $C_i$ are the Wilson coefficients. ($C_{7V}$ is proportional to $y_{7V}$ above).

G.D'Ambrosio, G.Ecker, G.Isidori and J.Portoles, hep-ph/9808289

- Phenomenological values: $a_+ = -0.578 \pm 0.016$ and $|a_S| = 1.06^{+0.26}_{-0.21}$.

Can we do better in lattice simulations?

- The generic non-local matrix elements which we wish to evaluate is

$$X \equiv \int_{-\infty}^{\infty} dt_x\, d^3x\, \langle \pi(p)\,|\,\mathrm{T}\,[J(0)\,H(x)]\,|K\rangle$$
$$= i\sum_n \frac{\langle \pi(p)\,|\,J(0)\,|n\rangle\,\langle n\,|H(0)\,|\,K\rangle}{m_K - E_n + i\varepsilon} - i\sum_{n_s} \frac{\langle \pi(p)\,|\,H(0)\,|n_s\rangle\,\langle n_s\,|J(0)\,|\,K\rangle}{E_{n_s} - E_\pi + i\varepsilon}\,,$$

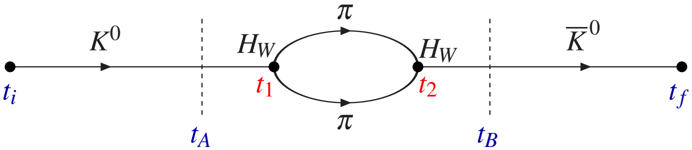
  - $\{|n\rangle\}$ and $\{|n_s\rangle\}$ represent complete sets of non-strange and strange sets.

- In Euclidean space we envisage calculating correlation functions of the form

$$C \equiv \int_{-T_a}^{T_b} dt_x\,\langle \phi_\pi(\vec{p},t_\pi)\,\mathrm{T}\,[J(0)\,H(t_x)]\,\phi_K^\dagger(t_K)\,\rangle \equiv \sqrt{Z_K}\,\frac{e^{-m_K|t_K|}}{2m_K}\,X_E\,\sqrt{Z_\pi}\,\frac{e^{-E_\pi t_\pi}}{2E_\pi}\,,$$

where

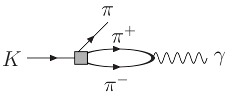
$$X_{E_-} = -\sum_n \frac{\langle \pi(p)\,|\,J(0)\,|n\rangle\,\langle n\,|H(0)\,|\,K\rangle}{m_K - E_n}\left(1 - e^{(m_K - E_n)T_a}\right) \quad \text{and}$$
$$X_{E_+} = \sum_{n_s} \frac{\langle \pi(p)\,|\,H(0)\,|n_s\rangle\,\langle n_s\,|J(0)\,|\,K\rangle}{E_{n_s} - E_\pi}\left(1 - e^{-(E_{n_s} - E_\pi)T_b}\right).$$

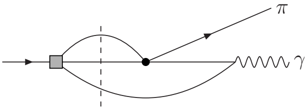



- In the $\Delta M_K$ computation, there is, of course, a two-pion intermediate state and we have had to control the corresponding finite-volume effects.
- This has been done on the assumption that the dominant intermediate states below $m_K$ are the two-pion states.

# Rescattering effects in rare kaon decays

- We can remove the single pion intermediate state.
- Which intermediate states contribute?
  - Are there any states below $M_K$?
  - We can control $q^2$ and stay below the two-pion threshold.




  - Do the symmetries protect us completely from two-pion intermediate states at low $q^2$?
  - Are the contributions from three-pion intermediate states negligible?
  - Answers to the above questions will affect what the finite-volume corrections are?
  - The ChPT-based phemomenology community neglect such possibilities as they are higher order in the chiral expansion.

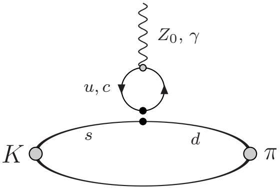
All to be investigated further!

- It looks as though the FV corrections are much simpler than for $\Delta M_K$ and may be exponentially small?

# Short Distance Effects

$$T_i^\mu = \int d^4x\, e^{-iq\cdot x}\, \langle \pi(p)\,|\,\mathrm{T}\{J^\mu(x)\,Q_i(0)\,\}\,|\,K(k)\rangle\,,$$

- Each of the two local $Q_i$ operators can be normalized in the standard way and for $J$ we imagine taking the conserved vector current.
- We must treat additional divergences as $x \to 0$.

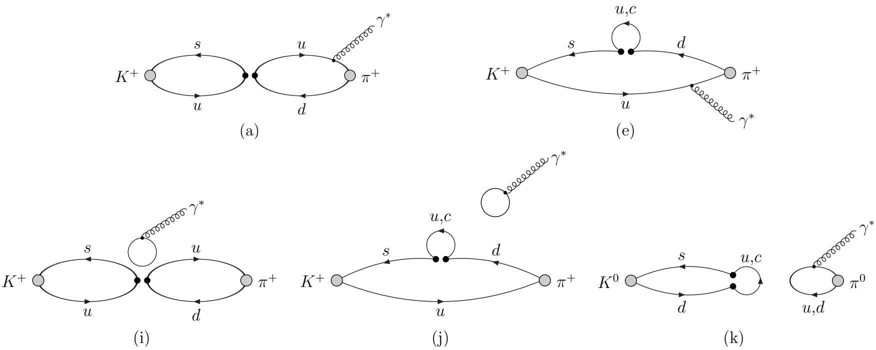



- Quadratic divergence is absent by gauge invariance $\Rightarrow$ Logarithmic divergence.
  - Checked explicitly for Wilson and Clover at one-loop order.
    G.Isidori, G.Martinelli and P.Turchetti, hep-lat/0506026
  - Absence of power divergences does not require GIM.
  - Logarithmic divergence cancelled by GIM.
  - For DWF the same applies for the axial current.

- To be investigated further!

## Many diagrams to evaluate!

- Sample diagrams:




- The top two diagrams are each representatives of four diagrams, corresponding to the four positions at which the photon can be inserted.
- The last diagram is only present for $K^0$ decays. The remaining diagrams are present for both $K^+$ and $K^0$ decays.
- The ChPT LECs are obtained at $q^2 = 0$, but we are not limited to this choice.

- We are now in a position to begin computing long-distance effects in weak processes, including $\Delta M_K$ and rare kaon decay amplitudes.
  - This builds on many years of theoretical developments (which continue) as well as improved algorithms and computing resources.
- To proceed most effectively we also need to build on the long experience of the non-lattice community.
- If we succeed, then perhaps we will also help to influence the setting of priorities for experimental particle physics in our joint exploration of the limits of the standard model and in searches for new physics.